Drawing complete outer-1-planar graphs in linear area

Therese Biedl*

Abstract

A *complete outer-1-planar graph* is a graph that can be drawn such that every edge has at most one crossing, all vertices are on the infinite face, and the so-called dual tree is a complete ternary tree. We show that every complete outer-1-planar graph has a straight-line grid-drawing that has area $O(n)$.

1 Introduction

In this paper we consider the question of how to create a *straight-line grid-drawing* of a graph, i.e., we want to map the vertices to grid points, and draw edges as straight-line segments between their endpoints such that vertex-points are distinct and no edge-segment contains a vertex-point except at its endpoints. If the input graph is *planar* (it has a *planar drawing* without crossing), then we further require that the drawing is likewise planar. Generally, whenever the given graph comes with a drawing (not necessarily using straight lines), then we expect the created straight-line grid-drawing to reflect the given drawing of the graph.

The objective is usually to achieve small *area of the drawing* (i.e., the area of the minimum enclosing axis-aligned bounding box of the drawing). Let n be the number of vertices. Any graph can be drawn with area $O(n^3)$ by placing the vertices on the moment-curve. For planar graphs, it has long been known that $O(n^2)$ is always sufficient [15, 16], and for some planar graphs $\Omega(n^2)$ area is required in a planar drawing [14]. For some subclasses of planar graphs, sub-quadratic area can be achieved. Of particular relevance to this paper are the results for *outer-planar graphs*, i.e., graphs that have a planar drawing where all vertices are incident with the unbounded region (the *outer-face*). Such graphs have straight-line grid-drawings in sub-quadratic area [9], and very recently the area has been reduced to $O(n^{1+\varepsilon})$ [13].

We are interested here in drawing *1-planar graphs*, i.e., graphs that have a drawing that is not necessarily planar but every edge is crossed at most once. Such graphs do not always have a straight-line grid-drawing [10] but if they are 3-connected then there is a straight-line drawing after deleting at most one edge [2] and the area is quadratic. Clearly some 1-planar graphs require $\Omega(n^2)$ area since all planar graphs are also 1-planar.

The natural question is now whether there are subclasses of 1-planar graphs that have straight-line grid-drawings in sub-quadratic area? The most obvious class to consider are *outer-1-planar graphs*, which are 1-planar graphs with a 1-planar drawing where all vertices are on the outer-face. It is known that outer-1-planar graphs can be drawn in sub-quadratic area in the drawing style of "visibility representations" (not reviewed here) [4]. Straight-line drawings of outer-1-planar graphs appear to have studied only a little bit. Dekhordi and Eades showed that they have so-called RAC-drawings [8] but they did not analyze the area. Auer et al. [3] showed that they have a straight-line grid-drawings in quadratic area. Bulatovic [5] achieved sub-quadratic area in some special situations.

In the pursuit of sub-quadratic-area drawings for outer-planar graphs [9, 13], one helpful ingredient was to first study a *complete outer-planar graph*, i.e., an outer-planar graph for which the dual graph (minus the outer-face vertex) is a complete binary tree when rooting it suitably. By exploiting its recursive structure, Di Battista and Frati showed that a complete outer-planar graph has a straight-line grid-drawing in $O(n)$ area [9].

In the same spirit, we ask here whether we can create small straight-line grid-drawings of *complete outer-1-planar graphs* (defined formally below). Bulatovic [5] showed that these have a grid-drawing of area $O(n^{2\cdot\log_3 2}) = O(n^{1.26})$. In this paper, we improve on this result and show that all complete outer-1-planar graphs have a straight-line grid-drawing of area $O(n)$. This fits into a long line of research of achieved optimal $O(n)$ area for straight-line grid-drawings of special graphs, see e.g. [1, 6, 7, 9].

2 Preliminaries

We assume familiarity with graph theory and planar graphs, see for example [11]. Assume throughout that G is an outer-1-planar graph with n vertices that is maximal in the sense that no edges can be added while maintaining simplicity and outer-1-planarity. Then G consists of an n-cycle as the outer-face and chords of the n-cycle. The *skeleton* G^s of G is the subgraph of G formed by the *uncrossed* edges, i.e., edges without crossing. The *inner faces* of G^s are the maximal bounded regions that contain no edges of G^s; it is known that

*David R. Cheriton School of Computer Science, University of Waterloo, biedl@uwaterloo.ca

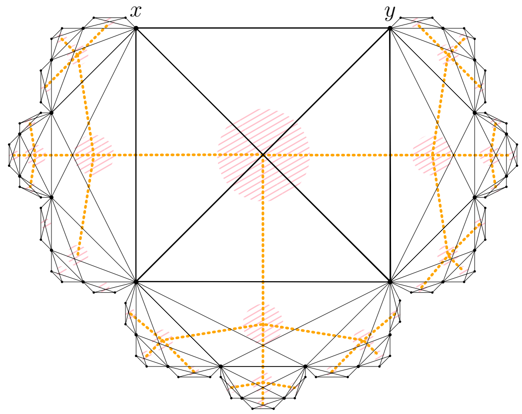

Figure 1: The complete outer-1-planar graph of depth 4. The dual tree is orange (striped/dotted).

all inner faces of G^s are triangles or quadrangles if G is maximal outer-1-planar [8]. The *dual tree* of G is obtained by creating a vertex for every inner face of G^s and making them adjacent if the corresponding faces share an edge. The dual tree of an outer-1-planar graph is (as the name suggests) a tree and all vertices have degree at most 4.

We call G a *complete outer-1-planar graph* if the dual tree $\mathcal{T}$ is a complete ternary tree after rooting it suitably. See Figure 1. The *depth* D of G is the number of vertices on the path in $\mathcal{T}$ from the root to the leaves. If $D \geq 2$, then G consists of K_4 (drawn with one crossing and corresponding to the root of the dual tree) with three copies of a complete outer-1-planar graph of depth $D-1$ attached at three of the four uncrossed edges of K_4. The *poles* of G are the endpoints of the uncrossed edge (x, y) of K_4 that is on the outer-face of G.

For an uncrossed edge (a, b) not on the outer-face, the *hanging subgraph* H_{ab} *at* (a, b) is the maximal subgraph that has (a, b) on the outer-face and does not contain both poles of G. The poles of H_{ab} are a and b.

The complete outer-1-planar graph of G depth D has $\Theta(3^D)$ vertices, hence $D \in \Theta(\log n)$. It is very easy to draw G in a grid of width $O(n)$ and height $O(D)$ [5], so with area $O(n \log n)$. But achieving linear area with this approach seems hopeless since even the skeleton of G requires $\Omega(\log n)$ width and height in any drawing. (This follows from [12] since its so-called pathwidth is logarithmic.) Instead for a linear-area drawing we construct a drawing of width and height $O(\sqrt{n})$.

Triangular grids. One ingredient for drawing complete outer-1-planar graph in linear area will be to use the grid points of a *triangular grid* (with grid-lines of slope $\sqrt{3}, 0, -\sqrt{3}$), rather than the standard (orthogonal) grid. This makes no difference overall, since the triangular grid can be mapped to an orthogonal grid with a shear, but allows us to treat hanging subgraphs symmetrically.

The following shortcuts will be useful. We use arrows such as $\nearrow$ and $\nwarrow$ for grid-lines of slope $\sqrt{3}$ and $-\sqrt{3}$, and so for example speak of a $\nearrow$-ray or the distance in $\nwarrow$-direction. An *axis-aligned equilateral triangle* is a triangle with three equal sides that all lie along grid-lines. An *axis-aligned isosceles triangle* is a triangle where two equal-length sides lie along grid-lines while the third side connects two grid points and has angle 30° on both ends. We will usually drop "axis-aligned" as we study no other equilateral or isosceles triangles. A triangle is called *upward* if it has a unique *top corner*, i.e., point with maximum y-coordinate. We use terms such as top/bottom/left/right side/corner only when this uniquely identifies the feature.

3 Drawing types

Let G be the complete outer-1-planar graph of depth D, and let x, y be its poles. We will need three kinds of drawings of G that will be combined recursively:

A *type-A* drawing $\mathcal{A}$ of G is contained within an equilateral upward triangle T. Vertices x and y are placed on the left and right side of T, respectively, with distance exactly D from the top corner. Drawing $\mathcal{A}$ occupies no points on the right side of T except for y. See Figure 2.

Furthermore, $\mathcal{A}$ must have the flexibility to move x as follows. Let the *wedge of* $\mathcal{A}$ be the smaller wedge between the $\nearrow$-ray and the $\nwarrow$-ray emanating from x. We require that for any position x' within the wedge, moving x to x' gives a drawing of G for which all edges are either within T or within the triangle spanned by x', y and the left corner of T.

A *type-B* drawing $\mathcal{B}$ of G is contained within an equilateral upward triangle T. Vertices x and y are placed at the top and right corner of T, respectively, and the left corner is empty. See Figure 2.

Furthermore, $\mathcal{B}$ must have the flexibility to move y as follows. Let z be the point on the bottom side of T that has distance exactly D to y (we call this the *attachment point* of $\mathcal{B}$). Let the *wedge of* $\mathcal{B}$ be the smaller wedge between the $\searrow$-ray and the $\rightarrow$-ray emanating from y. We require that for any position y' within the wedge, moving y to y' gives a drawing of G. Furthermore, the drawing is contained within T and the triangle spanned by x, y', z.

We call a type-B drawing a *type-B^+-drawing* if additionally no point other than x is on the left side of triangle T. With the exception of $D = 1$ all type-B drawings that we construct are actually type-B^+-drawings.

A *type-C* drawing $\mathcal{C}$ of G is contained within an isosceles upward triangle T where the left and bottom side have the same length. Vertices x and y are placed at the top and right corner of T, respectively. Drawing

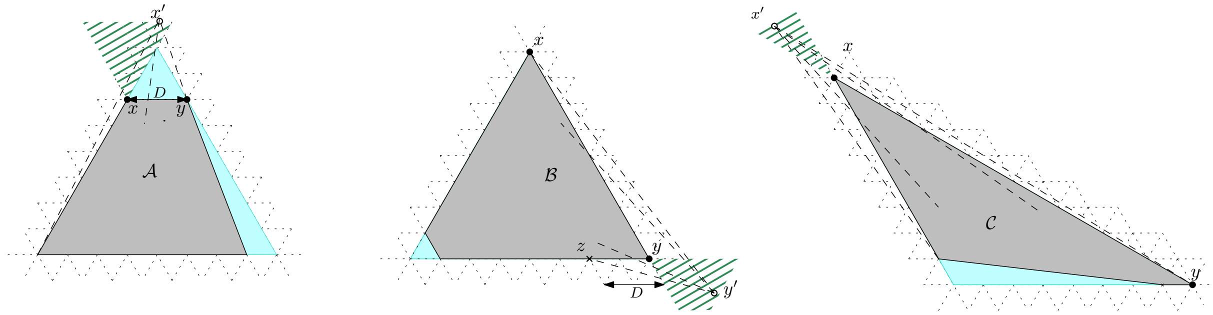

Figure 2: Drawings of type A, B and C. The wedge is green (striped).

$\mathcal{C}$ occupies no points on bottom side of T except for y and (possibly) points within unit distance from y. See Figure 2.

Furthermore, $\mathcal{C}$ must have the flexibility to move x as follows. Let the *wedge of* $\mathcal{C}$ be the smaller wedge between the $\nwarrow$-ray and a ray with slope $-\sqrt{3}/2$ (i.e., extending $\overline{xy}$) emanating from x. For any position x' within the wedge, moving x to x' gives a drawing of G.

Define the following function $w(\cdot)$ on positive integers:

$$w(D) := \begin{cases} 2 & \text{if } D = 1 \\ 6 & \text{if } D = 2 \\ 3w(D-2) + 4(D-2) + 6 & \text{if } D \geq 3 \end{cases}$$

A simple proof by induction shows that

$$w(D) \leq 16 \cdot 3^{D/2-1} - 2D - 5 \in O(3^{D/2}).$$

We will show the following by induction on D:

Lemma 1 *The complete outer-1-plane graph of depth D has drawings of type A, B and C where the shortest side of the bounding triangle T has length exactly $w(D)$. It also has a type-B^+ drawing where the side-length of T is at most $w(D) + 1$.*

In the base case (where $D = 1$ or 2) these drawings are easily created, see Figure 3 for some cases and Figure 10 in the appendix for all remaining ones.

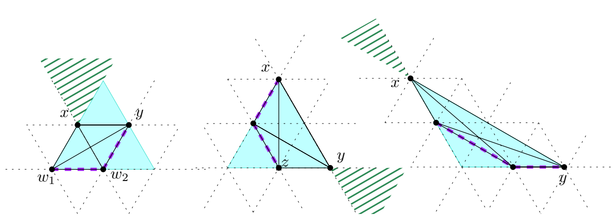

Figure 3: The drawings for $D = 1$ for type A, B, C.

4 The inductive step

Assume that the dual tree $\mathcal{T}$ of G has depth $D + 2$ where $D \geq 1$. We can hence split the graph into the subgraph Q corresponding to the root of $\mathcal{T}$ and and its three children, and the hanging subgraphs that are attached at the uncrossed edges that bound Q. (Each hanging subgraph is a complete outer-1-plane graph of depth D.) Enumerate the outer-face of Q as $\langle x, a, b, c, d, e, f, g, h, y \rangle$ in ccw order where x, y are the poles of G. See Figure 4.

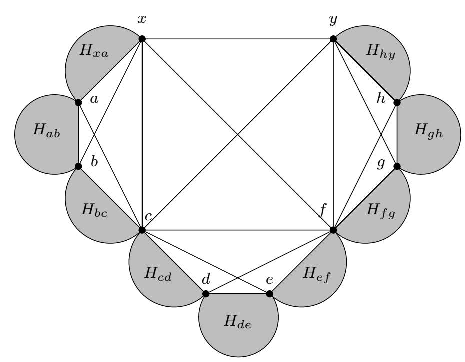

Figure 4: Splitting the graph into Q and nine hanging subgraphs.

The idea. Building a drawing of G uses the obvious recursive approach: create drawings of the nine hanging subgraphs of Q, combine them, and add the edges of Q. However, there are some intricate details with regards to placement of poles and spacing of subgraphs. We therefore first give a rough idea.

Observe that both an equilateral and an isosceles triangle T can be split into 9 equal-area triangles that are either equilateral or isosceles, see Figure 5. We assign the hanging subgraphs to these triangles as indicated in the figure, and plan to draw Q within the thick black lines (after expanding a bit).

Note that in our plan to place the vertices, some poles (e.g. vertex c for subgraph H_{bc}) are far away from the corresponding triangle; here the flexibility to move one pole within the wedge of the drawing will be crucial. However, this comes with the price that we must keep line segment $\overline{cz}$ free of other drawings, where z is the

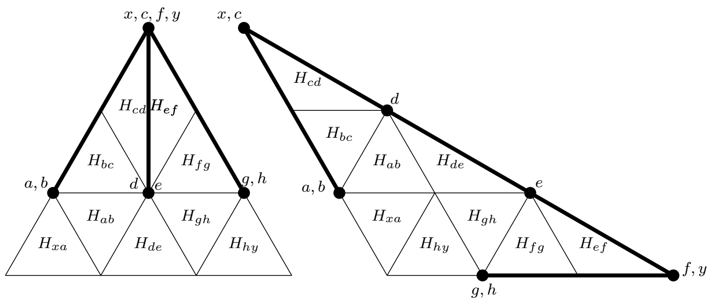

Figure 5: The idea of combining subgraphs. Locations for the vertices of Q are approximate.

attachment point of the drawing of H_{bc}. Therefore subgraphs cannot be placed exactly edge-to-edge as Figure 5 suggests and we must be more careful in spacing them.

Placing four subgraphs. We first explain how to place drawings of $H_{xa}, H_{ab}, H_{bc}, H_{cd}$; this will be the same for all three constructions below. Consult Figure 6. For any hanging subgraph H_{uv}, let Γ_{uv} be a (recursively obtained) drawing of H_{uv}—the text below will specify its type. Sometimes we will rotate Γ_{uv}; we use T_{uv} (drawn in cyan/light gray) for the bounding triangle of Γ_{uv} after such a rotation has been applied.

- Let Γ_{xa} be a type-A drawing for H_{xa}. The white circle in Figure 6 shows where pole x would be within Γ_{xa}, but it will actually be placed later somewhere within the wedge of Γ_{xa}.
- Let Γ_{ab} be a type-A drawing for H_{ab}, rotated by $+60°$. Place the left corner of T_{ab} one unit in ↘-direction from the top corner of T_{xa}. This puts pole a within the wedge of Γ_{ab} as required.
- Let Γ_{bc} be a type-B drawing for H_{bc}, rotated by $+120°$ and placed such that the two locations of b coincide. Pole c will be placed somewhere within the wedge of Γ_{bc}.
- Let Γ_{cd} be a type-C drawing for H_{cd}, rotated by $-60°$ and placed such that the left corner of T_{cd} coincides with the attachment point z of T_{bc}. Pole c will be placed somewhere within the wedge of Γ_{cd}.
- Consider the point where the ↗-ray from b intersects the ↑-ray from d, and let r_c be the ↗-ray emanating from here. We will later place c somewhere on ray r_c, which keeps it within both wedges of Γ_{bc} and Γ_{cd}, and keeps line segment $\overline{cz}$ outside all other drawings.

Observe that all drawings are disjoint except where they share a vertex. This holds because in a type-A drawing the right side only contains the pole, and in Γ_{cd} the shorter side at d contains points only within distance 1 from d, but these points are not used by Γ_{bc}. Also observe that for any placement of x within the wedge of Γ_{xa}, line segment $\overline{ax}$ will be outside all other drawings. Finally observe that the path a-b-c (shown thick dashed) is drawn with slopes alternating between $[0, \sqrt{3})$ and $\sqrt{3}$; this will be crucial below.

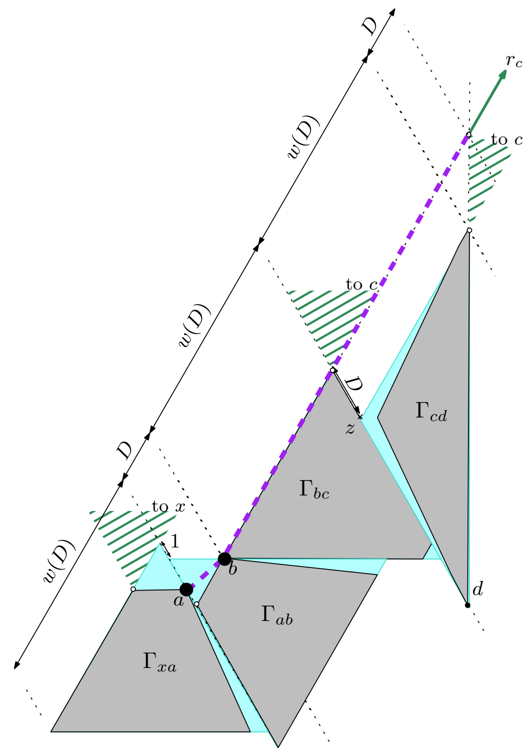

Figure 6: Placing $H_{xa}, \ldots, H_{cd}$.

Completing a type-A drawing. To complete the drawing to a type-A drawing, we copy and flip the existing drawing along a vertical line. See also Figure 7(a). More precisely, let ℓ_v be a vertical line that has →-distance $D/2$ from d. Mirror $\Gamma_{xa}, \ldots, \Gamma_{cd}$ along this line to get $\Gamma_{ef}, \ldots, \Gamma_{hy}$. The only subgraph missing is H_{de}, for which we use a type-A drawing that fits exactly with the existing points for d and e. One verifies that all drawings are disjoint except at common poles.

We define the bounding triangle T of the drawing to be the upward equilateral triangle that touches the left side of T_{xa}, has ↗-distance one to the bottom side of T_{de}, and has ↗-distance three from the right side of T_{hy}. (This is slightly asymmetric; the line ℓ_v does *not* go through the top corner of T.) Elementary computation shows that T has side-length $3w(D)+4D+6 = w(D+2)$ as desired. Place x and y (as required for a type-A drawing) at distance $D+2$ from the top corner of T; this puts x within the wedge of Γ_{xa}.

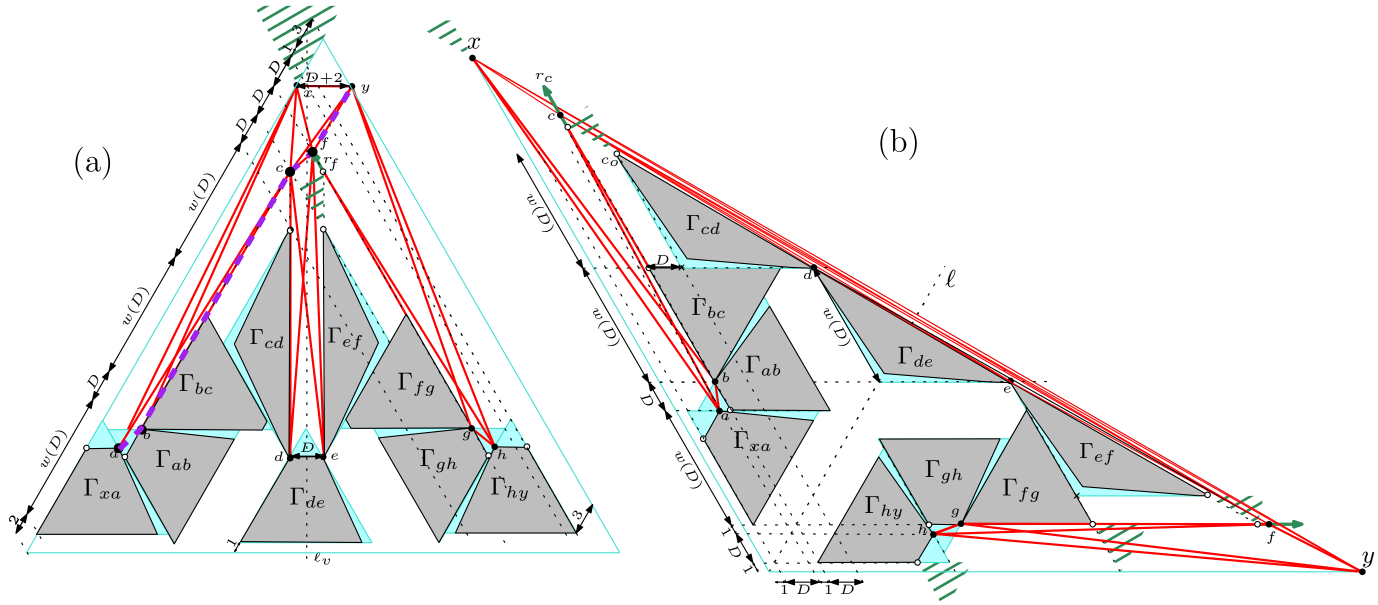

Figure 7: Creating (a) a type-A drawing and (b) a type-C drawing.

We place c at the start-point of ray r_c, which has ↖-distance $D{+}1$ from the left side of T. Let r_f be the copy of ray r_c on the right side; we place vertex f on this ray with ↖-distance $D{+}2$ from the left side of T. With this, $\overline{fy}$ has slope $\sqrt{3}$ while $\overline{cf}$ has slightly smaller slope.

We must argue that we have the flexibility to move x within the wedge $\mathcal{W}$ of the drawing. Consider the path $\pi = \langle w_1, w_2, \ldots, w_{2D+1}\rangle$ of neighbours of x. [The last five vertices on π are a, b, c, f, y, and this part is shown purple/dotted in Figures 3, 7, 10.] Path π connects the left side of T with the right side, and hence separates vertex x from all other vertices of the drawing. Also (as argued directly above or known by induction for the part of π in Γ_{xa}) the slopes along π alternate between a value in $[0, \sqrt{3})$ and exactly $\sqrt{3}$. For $1 \leq i \leq D$, let $\mathcal{W}_i$ be the smaller wedge between the two rays emanating from w_{2i} through w_{2i-1} and w_{2i+1}. By the slopes of the edges, $\mathcal{W}$ is strictly inside $\mathcal{W}_i$. Therefore $\{w_{2i-1}, w_{2i}, w_{2i+1}, x'\}$ forms a strictly convex quadrangle for any location of $x' \in \mathcal{W}$, and the K_4 formed by these four vertices is drawn with a crossing as required. Also, the quadrangles for different values of i are disjoint. So moving x' within $\mathcal{W}$ gives a drawing of G.

Creating a type-B drawing. To create a type-B drawing, we place all hanging subgraphs except H_{hy} exactly as in construction for the type-A drawing. Vertex h is placed as dictated by Γ_{gh}. For H_{hy} we use a type-B^+ drawing Γ_{hy} that we place such that the two drawings of h coincide. See Figure 8. One verifies that all drawings are disjoint except where they have common poles (this holds for Γ_{hy} since we use a type-B^+ drawing).

We define the bounding triangle T of the drawing to be the upward equilateral triangle that has ↖-distance one from the left side of T_{xa}, ↗-distance two from the line through $\overline{gh}$ and has side-length $3w(D) + 4D + 6 = w(D{+}2)$. Elementary computation shows that this triangle then includes Γ_{hy} since T_{hy} has side-length at most $w(D) + 1$. The left side of T is empty, so the created type-B drawing is automatically a type-B^+ drawing. We place x and y as required at the top and the right corner of T.

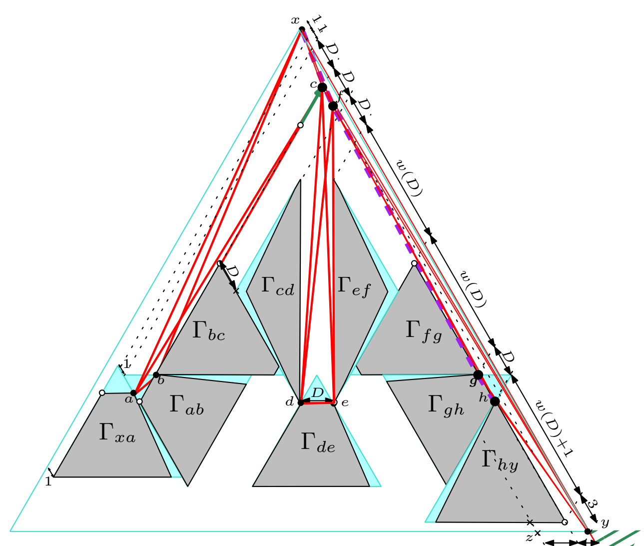

Figure 8: Creating a type-B drawing.

Let R be the right side of T. Place vertex c on r_c and vertex f on the ↑-ray from e, both with ↗-distance one to R. In particular $\overline{xy}$ is on R, $\overline{cf}$ has slope $-\sqrt{3}$ and ↗-distance one to R, and $\overline{gh}$ has slope $-\sqrt{3}$ and

↗-distance two to R; with this the complete graphs $\{x, y, c, f\}$ and $\{f, y, g, h\}$ of Q are drawn correctly (albeit with very small angles). All other edges of Q are also drawn correctly, see Figure 8. Also f is above e and to the right of the ↖-ray from g, hence within the wedges of Γ_{ef} and Γ_{fg} as required.

Also note that the attachment point of Γ_{hy} is the lowest point of the drawing, and its ↘-projection onto the bottom side of T has distance $D+2$ from y. Finally path x-c-f-g-h is drawn alternating between slopes in $[-\infty, -\sqrt{3})$ and $-\sqrt{3}$. Therefore as for type-A drawings one argues that y has the flexibility to move within its wedge, as long as nothing is placed between the attachment point z of T and the new location of y.

Creating a type-C drawing. Start with $\Gamma_{xa}, \ldots, \Gamma_{cd}$, placed as described above, but rotate everything by 60°. Let ℓ be the ↗-line that has ↘-distance $w(D)$ from d. Copy and mirror $\Gamma_{xa}, \ldots, \Gamma_{cd}$ along line ℓ to get $\Gamma_{ef}, \ldots, \Gamma_{hy}$. The only subgraph missing is then H_{de}, for which we use a type-C drawing that fits exactly with the existing points for d and e. See Figure 7(b). One verifies that all drawings are disjoint except where they have common poles.

We define the bounding triangle T to be the upward isosceles triangle where the left side is parallel to the left side of T_{xa} and at ↗-distance 1, the bottom side is parallel to the bottom side of T_{hy} and at ↗-distance 1, and the right side is parallel to the top side of T_{de} and at →-distance 2. (Line ℓ is the axis of symmetry for T.) Place x and y (as required for a type-C drawing) at the top and right corner of T. We place c and f on the rays r_c and r_f, with distance one from the start-point of the ray. This places the line through $\overline{cf}$ halfway between the line through $\overline{de}$ and the line through $\overline{xy}$. With this the complete graphs $\{x, y, c, f\}$ and $\{c, d, e, f\}$ of Q are drawn correctly (albeit with very small angles). All other edges of Q can clearly be added.

As for the flexibility of moving x, the same argument as for the type-A drawing applies with respect to the complete graph formed by $\{x, a, b, c\}$. For the complete graph formed by $\{x, c, f, y\}$, observe that $\overline{xy}$ and $\overline{cf}$ are parallel and therefore moving x to some point x' in the wedge (hence strictly above the line through $\overline{cf}$ keeps $\{x', c, f, y\}$ as a strictly convex quadrilateral.

To analyze the length of the shorter sides of T, let c_0 be the top corner of T_{cd}. Observe first (see also Figure 7(b)) that c_0 has ←-distance $2D+2$ to the left side of T and ↘-distance $3w(D)+2D+2$ to the bottom side of T. Now consider the close-up in Figure 9, let c_1 be the ←-projection of c_0 onto the left side of T, and let c_2 be the place where the line through $\overline{de}$ intersects the left side of T. Since $\overline{de}$ has slope $-\sqrt{3}/2$ while $\overline{c_0 c_1}$ has slope 0 and $\overline{c_1 c_2}$ has slope $-\sqrt{3}$, the triangle $\{c_0, c_1, c_2\}$ is isosceles, and therefore $d(c_1, c_2) = 2D+2$. The ↖-distance from c_2 to x is 2 by definition of T. Therefore the left side of T has length $3w(D)+4D+6 = w(D+2)$.

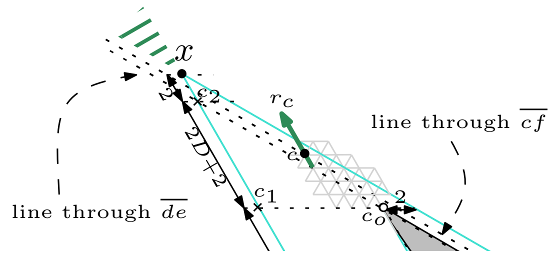

Figure 9: Close-up of the type-C construction.

This ends the proof of Lemma 1. Since a complete outer-1-planar graph has $n = 3^D + 1$ vertices, we have $w(D) \in \Theta(3^{D/2}) = \Theta(\sqrt{n})$ and the drawings reside (after a skew) in an orthogonal grid of area $O(n)$.

Theorem 2 *Every complete outer-1-plane graph has a straight-line drawing in a grid of $O(n)$ area.*

Following the steps of our construction, it is easy to construct the drawing in linear time.

5 Remarks

Our result is easily stated, but its proof is annoyingly complicated. The corresponding result for complete outer-planar graphs by Di Battista and Frati [9] has a very elegant proof: Draw a complete binary tree with a special property called "star-shaped", and one can derive a drawing of the balanced outer-planar graph from it. This does not translate to outer-1-planar graphs for multiple reasons. First, any complete outer-planar graph contains a complete binary tree (of roughly the same depth) as a subtree, so after drawing the complete binary tree one "only" has to add some edges. Attempts to generalize this for drawing a complete outer-1-planar graph G led to super-linear area [5]. The dual tree T of G is a complete ternary tree, but it does not map naturally to a subtree of G, and it would not be clear how to expand a drawing of T to one of G. Is there a simpler way to prove Theorem 2?

Also, in the paper by Di Battista and Frati [9] drawing the complete outer-planar graph was really just a warm-up to get results for all outer-planar graphs via star-shaped drawings of trees, useful also for [13]. We studied drawings of complete outer-1-planar graphs in the hopes that it would lead to sub-quadratic area-bounds for drawing *all* outer-1-planar graphs. But this seems significantly harder and obtaining area-bounds that are sub-quadratic (and ideally $O(n^{1+\varepsilon})$) remains open.

References

[1] H. Akitaya, M. Löffler, and I. Parada. How to fit a tree in a box. In *Graph Drawing and Network Visualization (GD'18)*, volume 11282 of *LNCS*, pages 361–367. Springer, 2018.

[2] M. J. Alam, F. J. Brandenburg, and S. G. Kobourov. Straight-line grid drawings of 3-connected 1-planar graphs. In *Graph Drawing (GD 2013)*, volume 8242 of *Lecture Notes in Computer Science*, pages 83–94. Springer, 2013.

[3] C. Auer, C. Bachmaier, F. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, and J. Reislhuber. Outer 1-planar graphs. *Algorithmica*, 74(4):1293–1320, 2016.

[4] T. Biedl, G. Liotta, J. Lynch, and F. Montecchiani. Optimal-area visibility representations of outer-1-planar graphs. In *Graph Drawing and Network Visualization (GD 2021)*, volume 12868 of *Lecture Notes in Computer Science*, pages 287–303. Springer, 2021.

[5] P. Bulatovic. Area-efficient drawings of outer-1-planar graphs. Master's thesis, University of Waterloo, September 2020.

[6] P. Crescenzi, G. D. Battista, and A. Piperno. A note on optimal area algorithms for upward drawings of binary trees. *Comput. Geom.*, 2:187–200, 1992.

[7] P. Crescenzi, P. Penna, and A. Piperno. Linear area upward drawings of AVL trees. *Comput. Geom.*, 9(1-2):25–42, 1998.

[8] H. R. Dehkordi and P. Eades. Every outer-1-plane graph has a right angle crossing drawing. *Int. J. Comput. Geom. Appl.*, 22(6):543–558, 2012.

[9] G. Di Battista and F. Frati. Small area drawings of outerplanar graphs. *Algorithmica*, 54(1):25–53, 2009.

[10] W. Didimo. Density of straight-line 1-planar graph drawings. *Inf. Process. Lett.*, 113(7):236–240, 2013.

[11] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[12] S. Felsner, G. Liotta, and S. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *J. Graph Alg. Appl*, 7(4):335–362, 2003.

[13] F. Frati, M. Patrignani, and V. Roselli. LR-drawings of ordered rooted binary trees and near-linear area drawings of outerplanar graphs. *J. Comput. Syst. Sci.*, 107:28–53, 2020.

[14] H. d. Fraysseix, J. Pach, and R. Pollack. Small sets supporting Fary embeddings of planar graphs. In *ACM Symposium on Theory of Computing (STOC '88)*, pages 426–433, 1988.

[15] H. d. Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[16] W. Schnyder. Embedding planar graphs on the grid. In *ACM-SIAM Symposium on Discrete Algorithms (SODA '90)*, pages 138–148, 1990.

Appendix

In Figure 10 we show the drawings for the base case in the other situations.

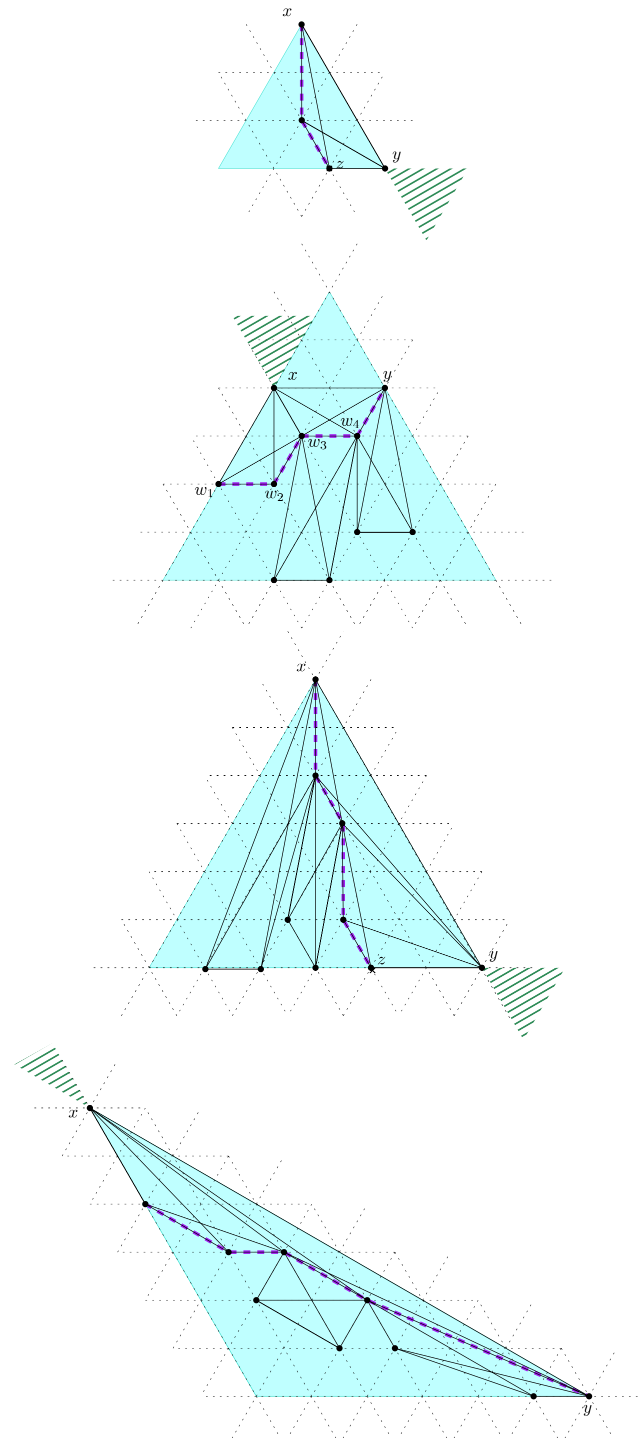

Figure 10: The type-B^+ drawing for $D = 1$ and the drawings (of type A, $B = B^+$ and C) for $D = 2$.