A $\frac{13}{9}$-approximation of the average-$\frac{2\pi}{3}$-MST

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Abstract

Motivated by the problem of orienting directional antennas in wireless communication networks, we study average bounded-angle minimum spanning trees. Let P be a set of points in the plane and let α be an angle. An α-spanning tree (α-ST) of P is a spanning tree of the complete Euclidean graph induced by P with the restriction that all edges incident to each point $p \in P$ lie in a wedge of angle α with apex p. An α-minimum spanning tree (α-MST) of P is an α-ST with minimum total edge length.

An average-α-spanning tree (denoted by $\overline{\alpha}$-ST) is a spanning tree with the relaxed condition that incident edges to all points lie in wedges with average angle α. An average-α-minimum spanning tree ($\overline{\alpha}$-MST) is an $\overline{\alpha}$-ST with minimum total edge length. In this paper, we focus on $\alpha = \frac{2\pi}{3}$. Let $A\left(\frac{2\pi}{3}\right)$ be the smallest ratio of the length of the $\overline{\frac{2\pi}{3}}$-MST to the length of the standard MST, over all sets of points in the plane. Biniaz, Bose, Lubiw, and Maheshwari (Algorithmica 2022) showed that $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$. In this paper we improve the upper bound and show that $A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

1 Introduction

A wireless communication network can be represented as a geometric graph in the plane. Each antenna is represented by a point p, its transmission range is represented by a disk with radius r centered at p, and there is an edge between two points if they are within each other's transmission ranges. The problems of assigning transmission ranges to antennas to achieve networks possessing certain properties has been widely studied [3, 5, 9, 12, 14, 15, 16, 17].

In recent years, there has been considerable research on the problem of replacing omni-directional antennas with directional antennas [1, 2, 4, 6, 8, 10, 11, 13, 14, 18]. Here, the transmission range of each point p is an oriented wedge with apex p and angle α. Directional antennas provide several advantages over omni-directional antennas, including less potential for interference, lower power consumption, and reduced area where communications could be maliciously intercepted [3, 18].

Motivated by this problem, Aschner and Katz [2] introduced the *α-Spanning Tree* (α-ST): a spanning tree of the complete Euclidean graph in the plane where all incident edges of each point p lie in a wedge of angle α with apex p. They also presented approximation algorithms for the cases where $\alpha = \frac{\pi}{2}, \frac{2\pi}{3}$, and π, with approximation factors of 16, 6, and 2, respectively, with respect to the MST. For $\alpha = \frac{2\pi}{3}$ and $\alpha = \frac{\pi}{2}$, the approximation ratios have been improved to $\frac{16}{3}$ [6] and 10 [7], respectively. Aschner and Katz further proved the NP-hardness of the problem of computing the α-MST for the $\alpha = \frac{2\pi}{3}$ and $\alpha = \pi$ cases.

Most previous research in this context has been done on the case where α is one fixed value for all antennas [6]. Biniaz et al. [6] extended this concept to an average-α-minimum spanning tree ($\overline{\alpha}$-MST): an α-MST with the relaxed restriction that the average angle of all the wedges is at most α. More formally, a total angle of αn must be allocated among n points p so that each point has a sufficient allowed angle to cover all incident edges. In the case where $\overline{\alpha} = \frac{2\pi}{3}$, they presented an algorithm that achieves an $\overline{\alpha}$-ST of length at most $\frac{3}{2}$ times the length of the MST. They also proved a lower bound of $\frac{4}{3}$ on the approximation factor with respect to the MST.

In this paper, we improve the upper bound on $A\left(\frac{2\pi}{3}\right)$ from $\frac{3}{2}$ to $\frac{13}{9}$. In fact we modify the algorithm of [6] and obtain an $\overline{\alpha}$-ST of length at most $\frac{13}{9}$ times the length of the MST. Our algorithm involves a stronger exploitation of the Euclidean metric than the previous work.

Our improved upper bound immediately gives an approximation algorithm with ratio $\frac{13}{9}$ (with respect to the MST) for the $\overline{\alpha}$-MST problem for any $\alpha \geq \frac{2\pi}{3}$. Similar to that of [6], our algorithm runs in linear time after computing the MST.

1.1 Notation

We use the terms point and vertex interchangeably depending on the context.

To facilitate comparison, we borrow the following notation from [6]. A *maximal path* in a tree is a path with at least two edges where all internal vertex degrees are 2, and the end vertex degrees are not 2. To *contract* a

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maximal path is to remove all vertices of degree 2 on the path and the edges between them, and add an edge connecting the end vertices. The angle that the incident edges of a vertex in an $\overline{\alpha}$-MST are allowed to fall within is called its *charge*. Charges can be redistributed between vertices. We denote the total length of edges of a geometric graph G by $w(G)$.

As the length of the optimal solution is not known, we use the underlying MST of the points as a lower bound in our analysis. We denote the smallest ratio of the length of the $\overline{\frac{2\pi}{3}}$-MST to the length of the standard MST over all points in the plane as $A\left(\frac{2\pi}{3}\right)$. In [6], it was shown that $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{3}{2}$.

1.2 Outline

The approximation algorithm of [6] for the $\overline{\frac{2\pi}{3}}$-MST starts with a standard MST that has maximum degree 5 (which always exists). Then it re-assigns angle charges from leaves to inner vertices. Their approach first considers the MST with all maximal paths contracted, and then introduces edge shortcuts in each contracted path.

By exploiting additional geometric properties we ensure the connectivity of path vertices with less total charge. This enables us to save some charges. The saved charges allow us to introduce fewer shortcuts than the original algorithm, resulting in a shorter $\overline{\frac{2\pi}{3}}$-ST.

2 The Algorithm of Biniaz et al.

In this section we briefly describe the algorithm of Biniaz et al. [6], which we refer to by "Algorithm 1".

The algorithm starts by computing a degree-5 minimum spanning tree T of the point set, where each vertex holds a charge of $\frac{2\pi}{3}$. Then the algorithm goes through two phases that redistribute the charges and also modify the tree. In the first phase, all maximal paths of T are contracted (to edges), resulting in a tree with no vertices of degree 2, and all other vertices having the same degree as in T. The charge from the leaves are then redistributed among the internal vertices so that each vertex of degree 3, 4, and 5 has a charge of $\frac{4\pi}{3}$, 2π, and $\frac{8\pi}{3}$, respectively. Since the charge of each internal vertex with degree n is at least $\left(1 - \frac{1}{n}\right) 2\pi$, which covers any set of n edges, all vertices can cover their incident edges. After redistribution, degree-1 vertices have 0 charge and each degree-2 vertex holds its original $\frac{2\pi}{3}$ charge. This redistribution retains a pool of $\frac{4\pi}{3}$ charge that can be split among all leaves in the tree at the end of the algorithm.

In the second phase, the edges of each path $p_1, p_2, \ldots, p_m$ that was contracted in phase 1 are split into two matchings, M_1 and M_2 with equal number of edges (if the path has odd number of edges then the last edge is not in either matching). The edges of the matching with the larger weight are removed, and a set $S = \{(p_1, p_3), (p_3, p_5), ...\}$ of new edges called *shortcuts* are introduced (see Figure 15 of [6], which we include here as Figure 1). By this process, the charge of every new degree-1 vertex is redistributed among other vertices so that each new degree-2 and degree-3 vertex along the path has a charge of π and $\frac{4\pi}{3}$, respectively; this is handled in four cases based on which matching is heavier and whether the path length is even or odd, as shown in Figure 1. Note that the charge given to vertices assigned degree 2 and 3 allows them to cover any set of 2 and 3 edges, respectively.

Let M_1' and M_2' be the union of the edges in the smaller and larger-weight matchings of all contracted paths, respectively. Let T' be the final tree obtained by the above algorithm, and let E be the set of edges of T not in $M_1' \cup M_2'$. Then $w(T) = w(E) + w(M_1') + w(M_2')$. By the triangle equality we have $w(S) \leq w(M_1') + w(M_2')$. Since $w(M_2') \geq w(M_1')$ we get

$$\begin{aligned} w(T') &= w(E) + w(M_1') + w(S) \\ &\leq w(E) + w(M_1') + w(M_1') + w(M_2') \\ &= w(T) + w(M_1') \leq \frac{3}{2} w(T). \end{aligned}$$

3 The Improved Algorithm

We begin by modifying the charge-redistribution of phase 2 of Algorithm 1 with a more careful charge redistribution. In particular we show that the 3 edges, that are incident to new degree-3 vertices, can be covered by $\frac{4\pi}{3} - \frac{\pi}{12}$ charge (meaning that we can *save* the $\frac{\pi}{12}$ charge). We then use the saved charge of $\frac{\pi}{12}$ to achieve a better approximation with respect to the original MST. The following lemma, although very simple, plays an important role in the design of the modified algorithm.

Lemma 1 *It is possible to save at least $\frac{\pi}{12}$ charge from every shortcut performed by phase 2 of Algorithm 1.*

Proof. Consider a shortcut ac between two consecutive edges ab and bc of a contracted path as depicted in Figure 2. Up to symmetry we may assume that ab is in M_2 and thus it has been removed in phase 2 of Algorithm 1. Denote the angle $\angle bca$ by β. Since the path (a, b, c) is part of the MST, ac is the largest edge of the triangle $\triangle abc$, and thus $\angle abc$ is its largest angle. Therefore $\beta \leq \frac{\pi}{2}$.

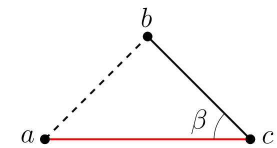

Figure 2: illustration of the proof of Lemma 1.

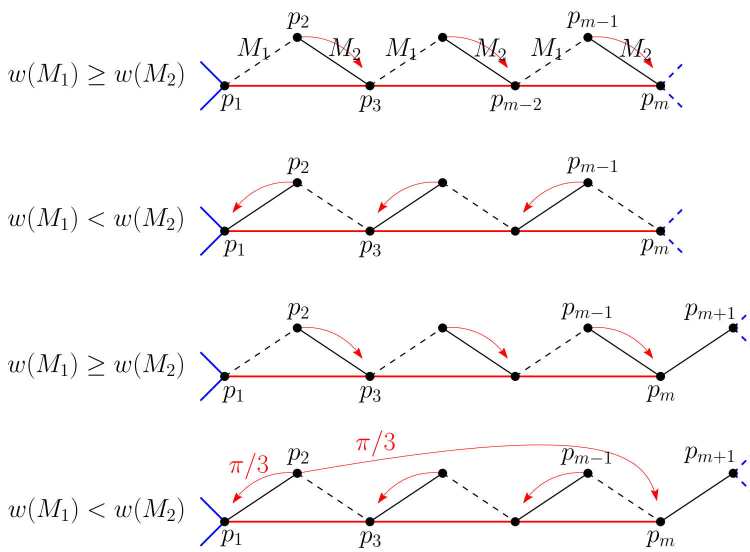

Figure 1: (borrowed from [6]) The contracted path is shown by black segments. The dashed-black edges belong to M_2 and the red edges belong to S.

The replacement of ab by the shortcut ac has not changed the degree of a, has decreased the degree of b by 1, and has increased the degree of c by 1. Thus the charge assigned to a by Algorithm 1 remains enough to cover its incident edges. Since b has degree 1, its $\frac{2\pi}{3}$ charge is free. Algorithm 1 transfers this free charge to c to cover its new edge. We show how to cover all edges incident to c while saving $\frac{\pi}{12}$ charge. If c's original degree (i.e. after phase 1 and before phase 2) was 4 or 5 then it carries at least 2π charge which is sufficient to cover its edges. We may assume that the original degree of c is 1, 2, or 3, in which case it holds a charge of 0, $\frac{2\pi}{3}$, or $\frac{4\pi}{3}$, respectively. Thus the new degree of c (after phase 2) is 2, 3, or 4. Based on this we distinguish three cases.

- If $\deg(c) = 2$ then the two incident edges of c are ac and bc. We can cover these edges by a charge of β ($\leq \frac{\pi}{2}$). Thus we transfer $\frac{\pi}{2}$ charge from b to c and we save $\frac{\pi}{6}$.
- If $\deg(c) = 3$ then we cover β and the smaller of the other two angles at c. Thus the three incident edges to c can be covered by charge of

$$\beta + \left(\frac{2\pi - \beta}{2}\right) = \frac{2\pi + \beta}{2} \leq \frac{2\pi + \frac{\pi}{2}}{2} = \frac{5\pi}{4}.$$

  Thus by transferring $\frac{7\pi}{12}$ from b to c it will have charge of $\frac{5\pi}{4}$ (including its original $\frac{2\pi}{3}$ charge). Thus we save charge of $\frac{2\pi}{3} - \frac{7\pi}{12} = \frac{\pi}{12}$ from b.
- If $\deg(c) = 4$ then we transfer $\frac{\pi}{6}$ charge from b to c and save the remaining $\frac{\pi}{2}$ charge of b. The vertex c now holds $\frac{3\pi}{2}$ charge (including its charge $\frac{4\pi}{3}$ after phase 1) which covers its four incident edges.

□

The following is a direct implication of Lemma 1.

Corollary 2 *It is possible to save $\frac{\pi}{3}$ charge from every four shortcuts that are performed by Algorithm 1.*

3.1 Reversing Shortcuts

In this section, we present an approximation algorithm that uses fewer shortcuts than Algorithm 1. In fact the new algorithm reverses a constant fraction of the shortcuts performed by Algorithm 1.

Theorem 3 *Given a set of n points in the plane and an angle $\alpha \geqslant \frac{2\pi}{3}$, there is an $\overline{\alpha}$-spanning tree of length at most $\frac{13}{9}$ times the length of the MST. Furthermore, there is an algorithm to find such an $\overline{\alpha}$-ST that runs in linear time after computing the MST.*

Proof. Let T be a degree-5 minimum spanning tree of the point set, and T' be the $\overline{\frac{2\pi}{3}}$-spanning tree obtained from T by Algorithm 1.

Consider the sequence of shortcuts introduced by Algorithm 1 along each contracted path. Let $s_1, s_2, ..., s_m$ be the concatenation of the sequences for all contracted paths. We split these shortcuts into nine sets $S_0, \dots, S_8$ such that $s_i \in S_{(i \mod 9)}$ for each $i \in \{1, \dots, m\}$. Note that no two adjacent shortcuts in the same contracted path will be in the same set S_i. Moreover the number of shortcuts in any two sets S_i and S_j differ by at most 1. Recall that the edges of each contracted path in Algorithm 1 are split into two matchings M_1 and M_2. Let M_1' be the set of edges that are kept in the tree (i.e. M_1' is the union of the smaller-weight matchings from each contracted path), and let the set of edges in the heavier matchings be M'_2. Among $S_0, \ldots, S_8$, let S_8 be the one whose corresponding edges in M'_1 have the largest total weight.

Our plan now is to reverse the shortcuts in S_8, i.e., to replace them by their corresponding edges in M'_2. Let S' be the union of $S_0, \ldots, S_7$. Notice that $|S'| \geq 8 \cdot (|S_8| - 1)$. Let C denote the pool of charges that is obtained after phase 1 of Algorithm 1, and recall that it contains $\frac{4\pi}{3}$ charge. For each shortcut in S' we reassign the charges between its corresponding points to save at least $\frac{\pi}{12}$ charge (as shown in Lemma 1), and add this charge to C. Thus the total charge of C is at least

$$\frac{4\pi}{3} + 8 \cdot (|S_8| - 1) \cdot \frac{\pi}{12} = (|S_8| + 1) \cdot \frac{2\pi}{3}.$$

We will show that to reverse each shortcut from S_8 it suffices to take $\frac{2\pi}{3}$ charge from C.

Consider any shortcut ac from S_8 between two consecutive edges ab and bc of a contracted path as depicted in Figure 3. We *reverse* this shortcut by replacing ac with the removed edge ab. We also reclaim any portion of b's charge that was transferred to c. Thus the reverse operation brings the charges of b and c back to what it was after phase 1 and before phase 2; in particular it brings the charge of b back to $\frac{2\pi}{3}$. There is one exceptional case where $w(M_1) < w(M_2)$ and the path has odd number of edges (the last case in Figure 1 where p_3, p_2, p_1 play the roles of a, b, c, respectively). In this case the charge of b (i.e. p_2) would be $\frac{\pi}{3}$ as p_m holds the other $\frac{\pi}{3}$ portion. (Since no two shortcuts in S_8 are adjacent in the same contracted path, we can analyze a reverse operation independently of others. Notice, however, that it is possible that two or more shortcuts of S_8 are adjacent at a vertex that has degree at least 3 after phase 1. In this case, the charge of such a vertex suffices to cover its edges after reversing the shortcuts since it will have at least $\frac{\pi}{6}$ charge added for each new edge introduced by the process described in Lemma 1.) The reverse operation does not change the degree of a and thus its charge remains sufficient to cover its edges. The reverse operation makes b of degree 2 and decreases the degree of c by 1.

We take $\frac{\pi}{3}$ charge from C for b to bring it to a charge of π, which covers its two incident edges. If $\deg(c) = 1$ or $\deg(c) \geq 3$, its charge is sufficient to cover its edges. If $\deg(c) = 2$ then we take an additional charge of $\frac{\pi}{3}$ from C for c to cover its two incident edges. In the exceptional where $w(M_1) < w(M_2)$ and the path has odd number of edges (the last case in Figure 1), $p_2 = b$ holds $\frac{\pi}{3}$ charge, so we take $\frac{2\pi}{3}$ from C for p_2 to cover its two incident edges. Since $p_1 = c$ is of degree 1 or at least 3 (as the contracted path is maximal), its charge (acquired after phase 1) is sufficient to cover its edges. Thus, in the worst case we take $\frac{2\pi}{3}$ from C to reverse every shortcut.

After reversing all shortcuts in S_8, the pool C is left with at least $\frac{2\pi}{3}$ charge which can be distributed among the leaves of the resulting tree.

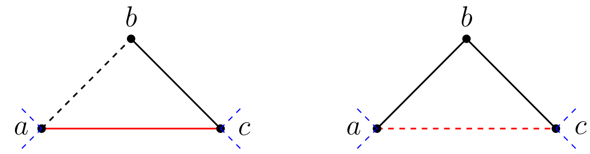

Figure 3: Left: The tree T' before reversing shortcut ac. Right: The tree T'' after reversing ac.

Let T'' be the $\overline{\frac{2\pi}{3}}$-ST tree obtained from T' after reversing all shortcuts in S_8. Let E be the set of edges of T'' not in $M'_1 \cup M'_2$. Let E' be the set of all edges of $M'_1 \cup M'_2$ that correspond to the shortcuts in S_8. Let $M''_1 = M'_1 \setminus E'$ and $M'_2 = M'_2 \setminus E'$ (i.e. all edges in M'_1 and M'_2, respectively, with a shortcut between their endpoints in T''). Then,

$$\begin{aligned} w(T'') &= w(E) + w(E') + w(S') + w(M''_1) \\ &\leq w(E) + w(E') + w(M''_1) + w(M''_2) + w(M''_1) \\ &= w(T) + w(M''_1). \end{aligned}$$

Since S_8 has the largest corresponding M'_1 weight, $w(M''_1) \leq \frac{8}{9} w(M'_1) \leq \frac{8}{9} \cdot \frac{1}{2} w(T) = \frac{4}{9} w(T)$. Thus,

$$w(T'') \leq w(T) + \frac{4}{9} w(T) = \frac{13}{9} w(T).$$

□

With Theorem 3 in hand, we report the following bound for $A\left(\frac{2\pi}{3}\right)$.

Corollary 4 $\frac{4}{3} \leq A\left(\frac{2\pi}{3}\right) \leq \frac{13}{9}$.

4 Conclusions

An obvious open problem is to further tighten the gap between the upper bound of $\frac{13}{9}$ and lower bound of $\frac{4}{3}$ for $\mathrm{A}\left(\frac{2\pi}{3}\right)$. This could be done by either introducing a new algorithm with a better approximation factor, or by finding a new set of points whose $\overline{\frac{2\pi}{3}}$-MST must have a weight of more than $\frac{4}{3}$ times that of the MST.

References

[1] E. Ackerman, T. Gelander, and R. Pinchasi. Ice-creams and wedge graphs. *Computational Geometry: Theory and Applications*, 46(3):213–218, 2013.

[2] R. Aschner and M. J. Katz. Bounded-angle spanning tree: Modeling networks with angular constraints. *Algorithmica*, 77(2):349–373, 2017.

[3] R. Aschner, M. J. Katz, and G. Morgenstern. Do directional antennas facilitate in reducing interferences? In *Proceedings of the 13th Scandinavian Symposium and Workshops on Algorithm Theory* (*SWAT*), pages 201–212, 2012.

[4] S. Ashur and M. J. Katz. A 4-approximation of the $\frac{2\pi}{3}$-MST. In *Proceedings of the 17th International Symposium on Algorithms and Data Structures (WADS)*, pages 129–143, 2021.

[5] A. Biniaz. Euclidean bottleneck bounded-degree spanning tree ratios. In *ACM-SIAM Symposium on Discrete Algorithms* (*SODA*), 2020.

[6] A. Biniaz, P. Bose, A. Lubiw, and A. Maheshwari. Bounded-angle minimum spanning trees. *Algorithmica*, 84(1):150–175, 2022.

[7] A. Biniaz, M. Daliri, and A. H. Moradpour. A 10-Approximation of the $\frac{\pi}{2}$-MST. In *Proceedings of the 39th International Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 13:1–13:15, 2022.

[8] P. Bose, P. Carmi, M. Damian, R. Y. Flatland, M. J. Katz, and A. Maheshwari. Switching to directional antennas with constant increase in radius and hop distance. *Algorithmica*, 69(2):397–409, 2014.

[9] P. M. Camerini. The min-max spanning tree problem and some extensions. *Information Processing Letters*, 7(1):10–14, 1978.

[10] I. Caragiannis, C. Kaklamanis, E. Kranakis, D. Krizanc, and A. Wiese. Communication in wireless networks with directional antennas. In *Proceedings of the 20th Annual ACM Symposium on Parallelism in Algorithms and Architectures* (*SPAA*), pages 344–351, 2008.

[11] P. Carmi, M. J. Katz, Z. Lotker, and A. Rosén. Connectivity guarantees for wireless networks with directional antennas. *Computational Geometry: Theory and Applications*, 44(9):477–485, 2011.

[12] T. M. Chan. Euclidean bounded-degree spanning tree ratios. *Discrete & Computational Geometry*, 32(2):177–194, 2004.

[13] M. Damian and R. Y. Flatland. Spanning properties of graphs induced by directional antennas. *Discrete Mathematics, Algorithms and Applications*, 5(3), 2013.

[14] S. Dobrev, E. Kranakis, D. Krizanc, J. Opatrny, O. M. Ponce, and L. Stacho. Strong connectivity in sensor networks with given number of directional antennae of bounded angle. *Discrete Mathematics, Algorithms and Applications*, 4(3), 2012.

[15] S. Dobrev, E. Kranakis, O. M. Ponce, and M. Plzík. Robust sensor range for constructing strongly connected spanning digraphs in UDGs. In *Proceedings of the 7th International Computer Science Symposium in Russia* (*CSR*), pages 112–124, 2012.

[16] M. M. Halldórsson and T. Tokuyama. Minimizing interference of a wireless ad-hoc network in a plane. *Theoretical Computer Science*, 402(1):29–42, 2008.

[17] S. Khuller, B. Raghavachari, and N. E. Young. Low-degree spanning trees of small weight. *SIAM Journal on Computing*, 25(2):355–368, 1996.

[18] E. Kranakis, F. MacQuarrie, and O. M. Ponce. Connectivity and stretch factor trade-offs in wireless sensor networks with directional antennae. *Theoretical Computer Science*, 590:55–72, 2015.