Quasi-Twisting Convex Polyhedra

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Abstract

We introduce a notion we call ***quasi-twisting*** that cuts a convex polyhedron P into two halves and re-glues the halves to form a different convex polyhedron. The cut is along a simple closed quasigeodesic. We initiate the study of the range of polyhedra produced by quasi-twisting P, and in particular, whether P can "quasi-twist flat," i.e., produce a flat, doubly-covered polygon. We establish a sufficient condition and some necessary conditions, which allow us to show that of the five Platonic solids, the tetrahedron, cube, and octahedron can quasi-twist flat. We conjecture that the dodecahedron and icosahedron cannot quasi-twist flat, and prove that they cannot under certain restrictions. Many open problems remain.

1 Introduction

A geodesic γ on a convex polyhedron P is a path that has exactly π surface angle to either side at every point of γ. So geodesics cannot pass through vertices. A ***quasigeodesic*** has at most π angle to each side at every point, and so can pass through vertices. Whereas most convex polyhedra have no simple closed geodesic [10], every convex polyhedron has at least three simple closed quasigeodesics, a result of Pogorelov from 1949 [16].

In this paper we introduce an operation we call ***quasi-twisting***, which applies to any convex polyhedron P and any simple closed quasigeodesic (or geodesic) Q on P. We imagine cutting along Q to separate P into two "halves" A and B, above and below Q, each with boundary $\partial A, \partial B$. Let L be the length of Q. Keeping B fixed, "glue" ∂A to ∂B, but shifted by some fraction of L. (A and B are considered flexible but isometric surfaces during this gluing.) So A is "twisted" with respect to B. By Alexandrov's Gluing Theorem, the result is a convex polyhedron $\tilde{P}$: the lengths of the boundaries ∂A, ∂B are equal, so the gluing results in a closed shape homeomorphic to a sphere. And because of the $\leq \pi$ quasigeodesic condition, both ∂A and ∂B are convex and so the re-gluing maintains $\leq 2\pi$ at every point along the seam.

Example. Before proceeding further, we illustrate with an example. P is a unit cube, and quasigeodesic Q is the path through six vertices illustrated in Fig. 1(a). (We identify a vertex either by its index i, or as v_i, whichever is more convenient.) The angles to one side of Q alternate between $\pi/2$ and π.

Cutting Q leaves A and B composed of three faces each, with vertex v_7 interior to A and antipodal vertex v_1 interior to B. The boundaries ∂A and ∂B each include a copy of the six vertices of Q. Now we twist A one unit counterclockwise from above, matching vertices of A to B as follows.

$$\begin{array}{cccccc} 2 & 6 & 5 & 8 & 4 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 6 & 5 & 8 & 4 \end{array}$$

Three of the six vertices along Q cease to be vertices in $\tilde{P}$. For example, $v_6 \to v_2$ joins π to π surface angle. The result is a triangular bipyramid with base an equilateral triangle of side length 2, and altitudes to v_1 and v_7 of length $\sqrt{2/3}$.

1.1 Related Work

Reshaping. Previous work on reshaping convex polyhedra relies on Alexandrov's Gluing Theorem [1, p.100] (which we abbreviate AGT):

Theorem AGT *If polygons are glued together satisfying three conditions:*

1. *All their perimeters are glued, without gaps or overlap.*
2. *At most 2π surface angle is glued at any point.*
3. *The result is homeomorphic to a sphere.*

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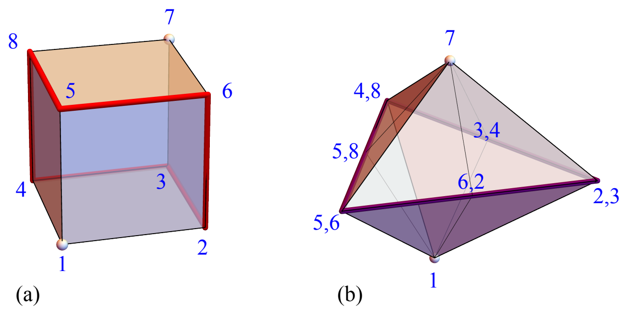

Figure 1: Vertices identified by their indices. (a) $Q = 265843$. (b) After quasi-twisting 1 unit: $5 \to 6, 6 \to 2, 2 \to 3$, etc.

Then the result is a convex polyhedron (possibly degenerated to a doubly-covered convex polygon), unique up to rigid motions.

A decade ago it was shown that every convex polyhedron could be unfolded to a single planar piece (possibly overlapping) and refolded to a different convex polyhedron [5]. A recent significant extension of this line of investigation showed (among other results) that any of the five Platonic solids can transform to any other by a sequence of at most six unfold-refold steps [4].

In a different direction, it was shown in [13] that, under mild conditions, a vertex can be excised from a convex polyhedron and transplanted elsewhere to create a new convex polyhedron. And [14] showed that any convex polyhedron can be converted to (a scaled copy) of any other via a sequence of vertex "tailorings"—excising a vertex along a digon and suturing the cut closed.

All of these reshaping results rely heavily on Alexandrov's Gluing Theorem, whose proof is nonconstructive. There is no practical algorithm for actually constructing the three-dimensional shape of the polyhedron guaranteed by AGT; only an impracticable pseudo-polynomial-time algorithm is available [11]. However, ad hoc calculations suffice for polyhedra with a few vertices (say, 8), or significant symmetries. And it seems possible that testing whether the polyhedron guaranteed by AGT is a doubly-covered polygon is easier than the general case, although one attempt in this direction did not achieve polynomial-time [12].

Quasigeodesics. Pogorelov's proof that there are always at least three simple closed quasigeodesics on a convex polyhedron is also non-constructive, and it has long been an open problem to design an algorithm to find a simple closed quasigeodesic (Open Problem 24.2 [7]). Recently there has been significant progress. First, a pseudopolynomial algorithm for finding at least one closed quasigeodesic (not necessarily simple) was detailed in [6]. Second, an exponential algorithm for finding all the simple closed quasigeodesics was described in [3]. Despite this progress, there remains no practical algorithm for finding simple closed quasigeodesics.

Questions. The quasi-twist operation suggests many questions, most fundamentally: From a given P, what quasi-twisted $\tilde{P}$ can result? A few remarks:

- Every P can be twisted to some $\tilde{P} \neq P$, because every P has simple closed quasigeodesics.
- $\tilde{P}$ could have as many as twice the number of vertices as P, and as few as half the number. For example, if P is a doubly-covered regular n-gon and Q its boundary, then quasi-twisting by angle π/n leads to $\tilde{P}$ with double the number of vertices. Reversing the roles of P and $\tilde{P}$ halves the number of vertices.
- Quasi-twisting P can lead to an uncountably infinite number of incongruent $\tilde{P}$. For example, quasi-twisting a doubly-covered n-gon by different angles in $(0, \pi/n)$ leads to incongruent $\tilde{P}$.

The regular n-gon example connects to *D-forms*, gluing two congruent convex shapes along their perimeters [7].

Since it is impractical both to find quasigeodesics and to apply AGT, algorithmic questions are currently unapproachable. Here we start the investigation of the natural question: Which P can be ***quasi-twisted flat***, i.e., is it possible to quasi-twist P to a doubly-covered polygon? We further narrow the question to the five Platonic solids. We show that the regular tetrahedron, the cube, and the regular octahedron can all quasi-twist flat. We conjecture that neither the dodecahedron nor the icosahedron can quasi-twist flat. Along the way, we establish some sufficient conditions for flattening by quasi-twists, and some necessary conditions, leaving complete characterization unresolved.

For brevity, henceforth we shorten "simple closed quasigeodesic" to ***quasigeodesic***, and "simple closed geodesic" to ***geodesic***. In contrast, a ***geodesic segment*** is a geodesic path between distinct vertices on P. A quasigeodesic is composed of geodesic segments.

2 Flattening with Perimeter Q

In the simplest examples of quasi-twisting to a doubly-covered polygon, the quasigeodesic Q becomes the perimeter of the doubly-covered polygon. We begin by giving necessary and sufficient conditions for this. Later we show that the regular tetrahedron and the cube satisfy these conditions, and in fact, we can even find a suitable Q in the 1-skeleton of P.

Lemma 1 *Quasi-twisting polyhedron P along a quasi-geodesic Q with twist τ produces a doubly-covered polygon whose perimeter is Q if and only if:*

1. *Q passes through every vertex of P;*
2. *At every point aligned by τ, the angles on the two sides of Q are equal.*

Proof. Suppose the conditions hold. Because Q includes every vertex of P, the interiors of the two sides A and B are empty of vertices, i.e., they are flat polygons. Because τ aligns points with equal angles (not only at the vertices of the polygon, but also along the sides), the two flat polygons are the same, so the result is a doubly-covered polygon.

For the other direction, if Q is the perimeter of a doubly-covered polygon, then Q must have passed through every vertex of P, and the twist τ has aligned equal angles. □

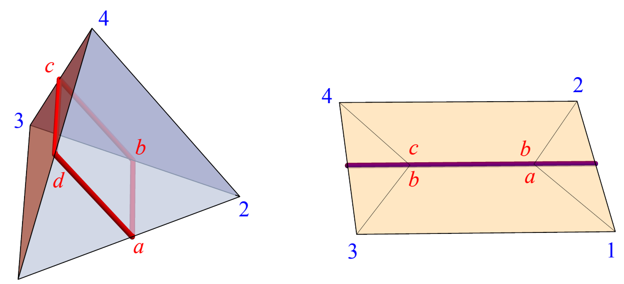

Figure 2: (a) $Q = abcd$. (b) Quasi-twisting results in a doubly-covered $1 \times \sqrt{3}/2$ rectangle.

This lemma is not, however, the only way to flatten P by quasi-twisting. We have several examples of P and Q that twist to flat polygons but which do not satisfy the conditions of the lemma. Perhaps the simplest is Q determined by the midplane of a regular tetrahedron, shown in Fig. 2(a). Here Q is a closed geodesic through no vertices, with two vertices to each side. If the edges are unit length, a twist by $\frac{1}{2}$, matching $abcd$ to $bcda$, results in a $1 \times \sqrt{3}/2$ doubly-covered rectangle, shown in (b). The four vertices become the corners of the rectangle and Q is not the boundary of the rectangle. We will show another example in Fig. 10.

3 Tetrahedron

The only quasigeodesic Q (up to relabeling) that includes all four vertices of a regular tetrahedron is shown in Fig. 3(a). Cutting Q partitions P into A and B, each alternating angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Quasi-twisting A one unit lines up the angles to match, as required by Lemma 1, resulting in a doubly-covered parallelogram, again alternating $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ angles.

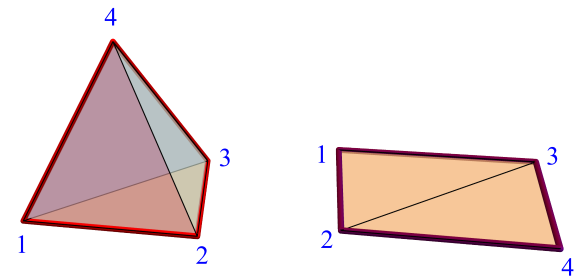

Figure 3: (a) Regular tetrahedron twists to (b) doubly-covered parallelogram.

4 Cube

We again follow Lemma 1. On a cube there is again one (up to relabelings) 8-vertex quasigeodesic, as shown in Fig. 4(a), alternating angles $\pi/2$ and π along Q. (The 3D shape of Q is sometimes known as a "napkin holder.") Quasi-twisting 2 units aligns the equal angles and results in the 3×1 doubly-covered rectangle shown in (b) of the figure: v_5, v_6, v_7, v_8 become flat with incident angle $\pi + \pi$, and the other four vertices have doubled angle $\pi/2$.

Using the same Q but quasi-twisting 1 unit results in a doubly-covered hexagon, where Q is not the boundary of the hexagon.

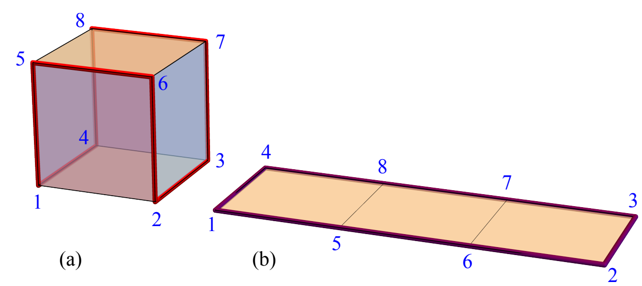

Figure 4: (a) $Q = 15623784$. (b) 3×1 doubly-covered rectangle.

5 Octahedron

Here we still use a quasigeodesic Q passing through every vertex of P, but we deviate from Lemma 1 in that we no longer align equal angles. The 6-vertex quasigeodesic Q shown in Fig. 5(a) has angles π times $\frac{1}{3}, \frac{2}{3}, 1, \frac{1}{3}, \frac{2}{3}, 1$. Fig. 5(b) shows that A and B each consists of four faces. Fig. 6(a) shows those faces unfolded, and (b) the result of shifting A by one unit. Gluing ∂A to ∂B after this shift results in creasing the layout as shown in (c), which folds to a doubly-covered $1 \times \sqrt{3}$ rectangle.

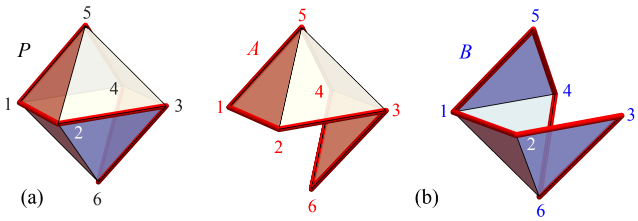

Figure 5: (a) $Q = 123645$. (b) A and B each consist of four faces.

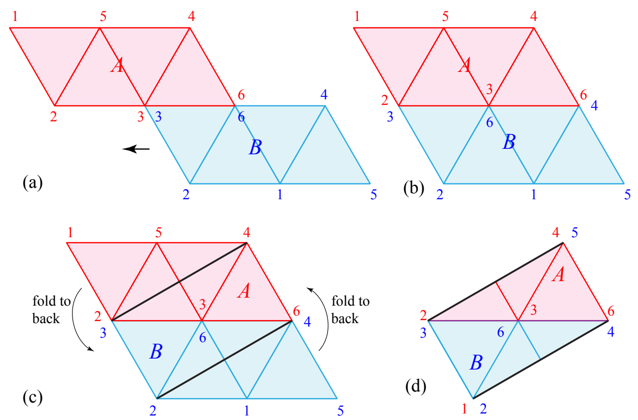

Figure 6: (a) A and B unfolded. (b) Shifting B one unit leftward. (c) Crease lines black. (d) Final doubly-covered $1 \times \sqrt{3}$ rectangle.

Quasi-twisting the different quasigeodesic shown in Fig. 7 by 1 unit leads to a doubly-covered equilateral triangle.

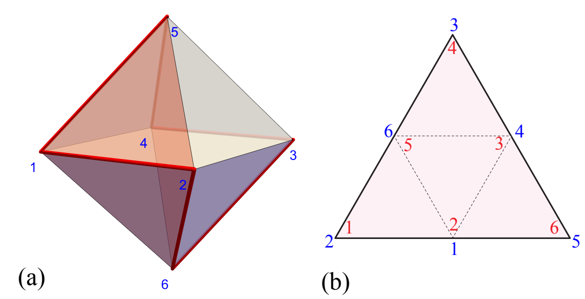

Figure 7: (a) Quasigeodesic $Q = 126345$. (b) Doubly-covered equilateral triangle, side length 2.

6 Dodecahedron and Icosahedron

We conjecture that neither the dodecahedron nor the icosahedron can quasi-twist to a doubly-covered polygon. We provide support for this conjecture by showing that the approach followed above—namely to use a quasigeodesic through all vertices—is not possible for the icosahedron or dodecahedron because no such quasigeodesics exist. We then discuss the challenges remaining to prove the conjecture, challenges that indicate what may be needed for a broader understanding of quasi-twists.

Lemma 2 *Any quasigeodesic Q on the icosahedron passes through at most 10 of the 12 vertices.*

Proof. Suppose Q passes through m vertices. Partition Q at the edges of the icosahedron into ***segments*** so that each segment is either: an edge of the icosahedron; a segment that crosses a face and is incident to one vertex of that face (we call these ***rays***); or a segment that crosses a face and is not incident to a vertex of that face. Suppose there are k edge segments, and r ray segments. Our counting argument need not include segments of the third type. First observe that Q consists of m vertex-to-vertex paths, each of which is an edge, or includes exactly two rays (one at either end of the path). Thus $m = k + \frac{1}{2}r$, so $r = 2m - 2k$.

Next, we claim (the argument is below) that each of the 20 triangle faces can contain at most one edge or ray segment. Since an edge is contained in two triangles, it counts twice. Thus $2k + r \leq 20$, and substituting $r = 2m - 2k$ gives $2m \leq 20$, so $m \leq 10$.

To prove the claim, suppose a face contains two edge or ray segments. If they are incident to the same vertex, then they must be consecutive on Q, and the angle between them is $\leq \frac{\pi}{3}$, leaving $\geq \frac{4\pi}{3}$ to the other side, violating the quasigeodesic condition. Otherwise (since a triangle does not have two vertex-disjoint edges), one segment must be a ray segment, say from vertex v to the opposite edge, and the other edge/ray segment must intersect it, see Figure 8(a). $\square$

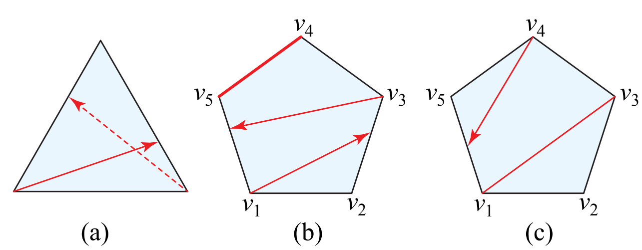

Figure 8: (a) Triangle of icosahedron. (b) Two rays and one edge in a dodecahedron face. (c) A chord v_1v_3 and a ray.

We now turn to the dodecahedron. The argument parallels that of the icosahedron.

Lemma 3 *Any quasigeodesic on the dodecahedron passes through at most* 18 *of the* 20 *vertices.*

Proof. Suppose Q passes through m vertices. We follow the same plan as in Lemma 2. Partition Q at the edges of the dodecahedron into *segments*. Here we have four types of segments: an edge of the dodecahedron; a segment that crosses a face from vertex to vertex (we call these ***chords***); a segment that crosses a face and is incident to one vertex (again ***rays***); or a segment that crosses a face and is not incident to any vertex of that face. Suppose there are k edge segments, c chord segments, and r ray segments. Again the counting argument need not include crossing segments, the last type. Henceforth we use ***segments*** to refer to edges, chords, and rays. Since Q consists of m vertex-to-vertex paths, each of which is an edge, a chord, or includes exactly two rays, we have $m = k+c+\frac{1}{2}r$ and thus $r = 2m-2k-2c$.

We make two claims about segments within a face to complete our counting argument.

- A face can contain three segments, but cannot contain four segments.
- If a face contains a chord then it has at most one other segment, and that other segment cannot be a chord.

See Fig. 8(b,c). These claims are proved below. The claims imply that each of the c chords is in a unique face with at most one other edge/ray segment, and the remaining $12-c$ faces each contain at most three edge/ray segments. Again, each edge segment counts twice since it lies in two faces. Thus $2k + r \leq c + 3(12 - c)$. Substituting for r we obtain $2m - 2c \leq c + 3(12 - c)$, which gives $2m \leq 36$, so $m \leq 18$.

To prove the claims, first suppose a face contains two segments incident to the same vertex. Then they must be consecutive on Q, and the angle between them is $\leq 108°$ leaving $\geq 216°$ to the other side, violating the quasigeodesic condition.

Next, consider the (disjoint) segments in the face. See Fig. 8(b). Each one cuts the face into two "sides," and we say that an "empty side" is a piece that contains no other segment. There are at least two empty sides and each (closed) empty side contains at least two vertices, leaving only one remaining vertex for a third (and last) segment. Thus there are at most three segments. There cannot be two (disjoint) chords, and if there is a chord, then one of its sides must be empty, and contains three vertices. Furthermore, a second empty side contains two vertices, so there cannot be a third segment. $\square$

We do not believe either Lemma 2 or Lemma 3 is tight, in that it seems neither 10- nor 18-vertex quasigeodesics are achievable on the icosahedron and dodecahedron respectively.

6.1 Conjecture Revisited

Having eliminated the possibility of using Lemma 1, the dodecahedron and the icosahedron could only twist to a flat polygon if Q does not include all the vertices. Then the vertices not on Q must flatten to the boundary of the doubly-covered convex polygon.

Again we use $\tilde{P}$ to represent P after quasi-twisting flat, and $\tilde{Q}$ to represent the image of Q on $\tilde{P}$. Thus $\tilde{Q}$ is a quasigeodesic on $\tilde{P}$. Note that $\partial\tilde{P}$ is itself a quasigeodesic of $\tilde{P}$ and indeed a *straightest* such quasigeodesic in the sense that it bisects the angle at each vertex through which it passes.[1]

If Q does not pass through all the vertices of P, then, by Lemma 1, $\tilde{Q}$ is different from $\partial\tilde{P}$, though they may share edges (see for example Figs. 2 and 6). It is tempting to imagine that there will be only two intersections between $\tilde{Q}$ and $\partial\tilde{P}$ as in those examples, in which case the two sections of $\partial\tilde{P}$ would lift to straightest quasigeodesic paths on A and B. However, the following example shows that there may be more than two intersections.

Let P be the doubly-covered rectangle shown in Fig. 9(a), and Q the horizontal bisector on the front and back, with v_1, v_3 above Q in A and v_2, v_4 below in B. P is already flattened, but we can still quasi-twist. Quasi-twisting by $\sqrt{2}/2$ leads to the doubly-covered unit square shown in (b). Directing $\tilde{Q}$ as $wxyz$, we have v_1, v_3 left of Q and v_2, v_4 right. This is not surprising, as the interiors of A and B are unaffected by quasi-twisting and gluing along their boundaries. What is perhaps surprising is that $\partial\tilde{P}$ is partitioned into four sections by $\tilde{Q}$, not the two sections one might expect.

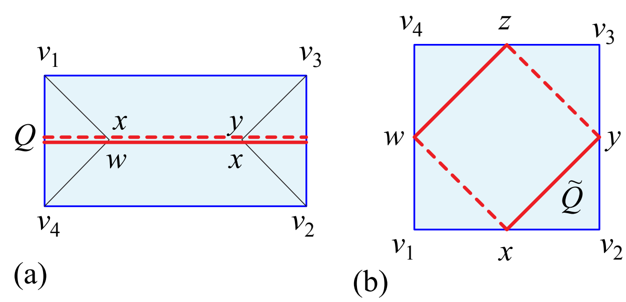

Figure 9: (a) $\sqrt{2} \times \sqrt{2}/2$ doubly-covered rectangle. (b) 1×1 doubly-covered square. Dashed portion of Q lie on the back side.

At the other extreme, one might ask if P can quasi-twist flat using a geodesic, i.e., one that does not pass through any vertices of P. This is possible for the tetrahedron (Fig. 2), and also for the octahedron (Fig. 10). There are three different geodesics on the cube [9, 8], but our experimental results show that none of them permit flat quasi-twisting. We conjecture that the icosahedron and dodecahedron cannot be twisted flat at true geodesics either. Fuchs and Fuchs [9] categorize the three possible geodesics on the icosahedron, but the case of the dodecahedron is apparently unresolved.

[1]The term "straightest geodesic" is from [17].

To summarize, we do not know if the icosahedron or dodecadedron can quasi-twist flat. A main difficulty is that we lack an understanding of when a quasigeodesic allows a flat quasi-twisting. Another impediment is that there is no complete inventory of the (simple) quasigeodesics on the dodecahedron or icosahedron. Just recently a 1-vertex quasigeodesic on the dodecahedron was found [2]. Even tetrahedra can have as many as 34 incongruent quasigeodesics [15].

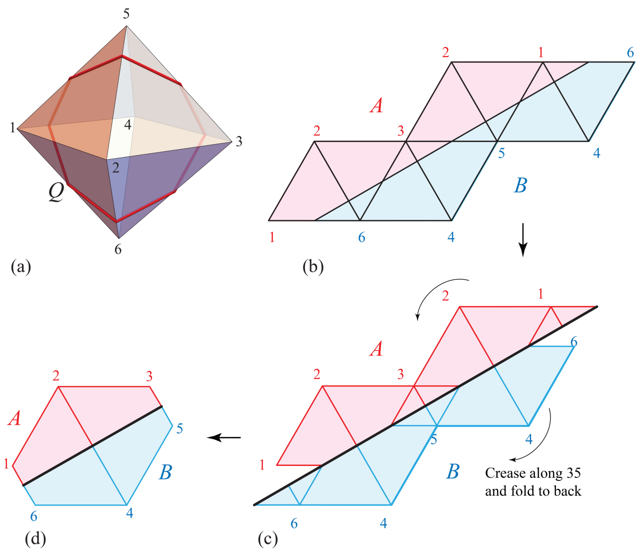

Figure 10: (a) Geodesic on octahedron. (b) A and B unfolded. (c) After quasi-twisting by $\sqrt{3}/2$. (d) Doubly-covered hexagon.

7 Open Problems

Because the quasi-twisting concept is new, almost every question one could pose is open. It would be interesting to know which polyhedra can be obtained from P by repeated quasi-twisting. Finding more substantive necessary conditions for quasi-twisting flat could resolve flattening the Platonic solids.

We emphasized quasi-twisting from P to a flat polyhedron $\tilde{P}$. The reverse viewpoint is equally interesting. We mentioned in Section 1 that a doubly-covered regular n-gon could be viewed as a discrete version of a D-form. It is natural to explore what shapes can be quasi-twisted from doubly-covered convex polygons. Even restricting to doubly-covered rectangles is interesting. For example, Fig. 11 illustrates quasi-twisting a doubly-covered square using the perimeter as the quasi-geodesic.

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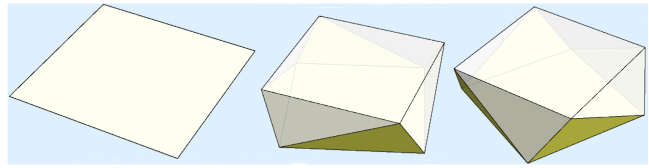

Figure 11: Snapshots of quasi-quasi-twisting a doubly-covered square. See `https://klaramundilova.com/square-twist/`.

References

[1] A. D. Alexandrov. *Convex Polyhedra*. Springer-Verlag, Berlin, 2005. Monographs in Mathematics. Translation of the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze, and A. B. Sossinsky.

[2] J. S. Athreya and D. Aulicino. A trajectory from a vertex to itself on the dodecahedron. *Amer. Math. Monthly*, 126(2):161–162, 2019.

[3] J. Chartier and A. de Mesmay. Finding weakly simple closed quasigeodesics on polyhedral spheres. In *38th Symp. Comput. Geom. (SoCG)*, Leibniz Internat. Proc. Informatics (LIPIcs), pages 27:1–27:16, June 2022.

[4] E. D. Demaine, M. L. Demaine, Y. Diomidov, T. Kamata, R. Uehara, and H. A. Zhang. Any regular polyhedron can transform to another by $o(1)$ refoldings. In *Proc. 33rd Canad. Conf. Comput. Geom.*, pages 332–342, 2021.

[5] E. D. Demaine, M. L. Demaine, J. i Itoh, A. Lubiw, C. Nara, and J. O'Rourke. Refold rigidity of convex polyhedra. In *28th European Workshop Comput. Geom. (EuroCG)*, pages 273–276, Mar. 2012.

[6] E. D. Demaine, A. C. Hesterberg, and J. S. Ku. Finding closed quasigeodesics on convex polyhedra. *Discrete Comput. Geom.*, 2021. arXiv:2008.00589. SoCG2020. To appear *Discrete Comput. Geom.*

[7] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007. `http://www.gfalop.org`.

[8] D. Fuchs. Periodic billiard trajectories in regular polygons and closed geodesics on regular polyhedra. *Geometriae Dedicata*, 170(1):319–333, 2014.

[9] D. B. Fuchs and E. Fuchs. Closed geodesics on regular polyhedra. *Moscow Mathematical Journal*, 7(2):265–279, 2007.

[10] P. Gruber. A typical convex surface contains no closed geodesic. *J. Reine Angew. Math.*, 416:195–205, 1991.

[11] D. Kane, G. N. Price, and E. D. Demaine. A pseudopolynomial algorithm for Alexandrov's theorem. In *Workshop Algorithms Data Struct. (WADS)*, pages 435–446. Springer, 2009.

[12] J. O'Rourke. On flat polyhedra deriving from Alexandrov's theorem. arXiv:1007.2016v2 [cs.CG]: `arxiv.org/abs/1007.2016v2`, July 2010.

[13] J. O'Rourke. Vertex-transplants on a convex polyhedron. In *Proc. 32nd Canad. Conf. Comput. Geom.*, pages 138–143, Aug. 2020.

[14] J. O'Rourke and C. Vîlcu. Reshaping Convex Polyhedra. arXiv 2107.03153v2 [math.MG]: `arxiv.org/abs/2107.03153`, July 2021.

[15] J. O'Rourke and C. Vîlcu. Simple closed quasigeodesics on tetrahedra. *Information*, 13:238–258, May 2022.

[16] A. V. Pogorelov. Quasi-geodesic lines on a convex surface. *Mat. Sb.*, 25(62):275–306, 1949. English transl., *Amer. Math. Soc. Transl.* 74, 1952.

[17] K. Polthier and M. Schmies. Straightest geodesics on polyhedral surfaces. In *ACM SIGGRAPH 2006 Courses*, pages 30–38. 2006.