Diverse Non Crossing Matchings*

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Abstract

A perfect matching M on a set P of n points is a collection of line segments with endpoints from P such that every point belongs to exactly one segment. A matching is *non-crossing* if the line segments do not cross. Two matchings M and N are said to be *compatible* if there are no crossings among any pair of line segments in $M \cup N$. We introduce a notion of diverse non-crossing matchings: a pair of perfect matchings M and N are k-diverse if, for every $p \in P$, the distance between the matched partners of p in M and N is at least k. In this contribution, we describe a polynomial time algorithm to determine if a set of points in convex position admits two compatible and perfect NCMs that are k-diverse. For points in convex position, we also show that if a perfect matching M is given as input, then we can determine, in polynomial time, if another perfect matching N exists that is compatible with M and is such that M and N are k-diverse. Finally, we also establish that every point set in general position admits a pair of compatible and perfect NCMs. The first two results also hold for bichromatic points, and we also give a characterization for when a bichromatic point set in convex position admits a pair of perfect and disjoint NCMs.

1 Introduction

Matching problems involve partitioning a set of objects into pairs subject to some constraints. For example, in the context of graphs, we are given a binary relation over the set of objects and require the pairs to be related. In a geometric setting, the set typically consists of geometric objects (Aloupis et al., 2013), and such problems have received a lot of attention because of their practical relevance.

Our focus is on the setting of matching points using line segments. In particular, given a set P of n points in the plane $\mathbb{R}^2$, we are interested in matching them with straight line segments. We focus on perfect non-crossing matchings (NCMs), i.e, matchings where every point is matched and no two line segments cross. Unless otherwise mentioned, we assume that all matchings are perfect.

It turns out that any collection of points admits a NCM and that this can be found in $O(n \log n)$ time (Hershberger and Suri, 1990; Lo et al., 1994). Many studies on NCMs focus on optimizing some structural property of the matching, such as the maximum, minimum, or average edge length. Two NCMs M and N are said to be *disjoint* if every point has a different matched partner in both matchings, and *compatible* if the segments in $M \cup N$ do not cross.

For optimization problems, the decision or search version of the question seeks to find *some* optimal solution, while the counting version asks to enumerate *all* optimal solutions. In many application scenarios, the former is not sufficient, while the latter is too demanding in terms of computational expense. This motivates the notion of demanding not all but a select collection of solutions. In most applications, the requirement is not just for a multitude of solutions, but for an "interesting" collection of solutions: for example, informally speaking, solutions that are minor variations of one another and are very similar may not be very useful in most settings.

The existence of a *diverse* collection of solutions has been explored in several settings recently. Studies on diverse solutions have focused on a wide array of problems including, but are not limited to, vertex cover (Baste et al., 2022), matchings (Fomin et al., 2020), stable matchings (Ganesh et al., 2021), matroids (Fomin et al., 2021), satisfiability (Nadel, 2011), Kemeny rank aggregation (Arrighi et al., 2021), etc.

To propose that we find "diverse" solutions, we need a notion of distance between solutions. In the setting of matchings between points in $\mathbb{R}^2$, a natural notion of "distance between matchings M and N" would be an aggregation of the distance between the matched partners of all the points in the two matchings. The aggregation function that we work with picks out the smallest such distance. In particular, using $M(\cdot)$ to denote the matched partner of a point p in a matching M, we define the distance between two matchings M and N over a point set P as $\min_{p \in P} d(M(p), N(p))$. Note that

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the we have used the term "distance" informally and this function does not satisfy the triangle inequality. We say that a collection of matchings $\mathcal{M}$ is k-diverse for some positive number k if the distance between every pair of matchings in $\mathcal{M}$ has a distance of at least k between them. Throughout our discussions, we focus on the problem of finding **two** matchings.

Our Contributions. We propose the following natural computational questions:

DIVERSE NCMs
(DIVERSE COMPATIBLE NCMs)

Input: A set P of $2n$ points and a positive rational number k.
Question: Does P admit two perfect matchings that are k-diverse and compatible, i.e., two DC-NCM's?

ANOTHER DIVERSE NCM
(ANOTHER DIVERSE COMPATIBLE NCM)

Input: A set P of $2n$ points, a perfect matching M over P, and a positive rational number k.
Question: Is there a perfect NCM N over P such that M and N are k-diverse (and compatible)?

We first show that any monochromatic point set P in general position with an even number of points such that $|P| \geqslant 4$ admits two compatible NCMs. Note that this is easy to see for points in convex position: a set of alternating edges on the convex hull and the remaining edges of the convex hull form a pair of compatible matchings. For points in general position, we generalize this idea by considering the layer decomposition and peeling off convex layers with an even number of points, and carefully matching across layers when we encounter layers with an odd number of points. We also characterize bichromatic point sets in convex position that admit two disjoint non-crossing matchings[1].

Theorem 1 (Disjoint Matchings) *Any point set P in general position admits two compatible perfect NCMs. A bichromatic point set P in convex position admits two disjoint and perfect NCMs if and only if the orbit of each point contains at least two points of the opposite colour.*

We next propose the computational problem of finding a matching that is diverse with respect to and, optionally, compatible with a given matching. We show that when points are in convex position, we can find such a matching in polynomial time. We use a dynamic programming approach here, considering subproblems corresponding to contiguous subintervals of the convex hull.

Theorem 2 (Another Diverse Matching) *For both monochromatic and bichromatic points in convex position, the problems* ANOTHER DIVERSE NCM *and* ANOTHER DIVERSE COMPATIBLE NCM *admit polynomial time algorithms.*

Finally, we consider the problem of finding a pair of diverse and compatible matchings. We demonstrate a polynomial time algorithm for points in convex position. For this algorithm, we note that any solution can be viewed equivalently as a collection of disjoint non-overlapping polygons. We prove a structural lemma which shows that there always exists an optimal solution consisting of polygons with a constant number of sides. We can then leverage this to come up with a dynamic programming algorithm that considers, as before, subproblems corresponding to contiguous subintervals of the convex hull, and makes progress by guessing all possible choices for the polygon that the first point on the subinterval belongs to.

Theorem 3 (Diverse Compatible Matching) *For both monochromatic and bichromatic points in convex position,* DIVERSE COMPATIBLE NCMs *admits a polynomial time algorithm.*

Related Work. The task of finding a matching that minimizes the length of the longest edge is called the bottleneck NCM problem and is known to be NP-complete in general and tractable for points in convex position and other special cases, and has been well-studied for monochromatic and bichromatic points (Abu-Affash et al., 2014; Carlsson et al., 2015; Savić and Stojaković, 2017, 2022; Biniaz et al., 2014).

Other variants of the problem such as those which involve minimizing the length of the shortest edge or maximizing the length of the longest edge are tractable (Mantas et al., 2021). Finally, to the best of our knowledge, the complexity of finding a matching that maximizes the length of the shortest edge is open.

In the context of the setting where we have a point set and a matching, it was conjectured by Aichholzer et al. (2009) that for every perfect matching M of a point set P such that $|P|$ is a multiple of four, there is another perfect matching, N of P such that M and N are compatible. This was subsequently proved by Ishaque et al. (2012) using a constructive argument that also leads to an efficient method for constructing the matching N. It is also known that the conjecture does not hold when $|P|$ is not a multiple of four Aichholzer et al. (2009).

[1]We refer the reader to Section 2 for the formal definitions of the terminology used here.

Organization of the paper. Due to lack of space, we defer the proofs of Theorem 1 and Theorem 2 to the full version of the paper. We provide most of the details towards showing Theorem 3 in Section 3, only deferring the argument of correctness and remarks about the bichromatic case to the full version.

2 Preliminaries

In the setting of monochromatic points, we use P typically to denote a set of $2n$ points in $\mathbb{R}^2$ with $n > 1$. When we work with points in general position, we will use $\mathcal{P}$ to denote the convex hull of P. In case of convex point sets, we label the points of P by $p_0, p_1, \ldots, p_{2n-1}$ in positive (counterclockwise) direction around the convex hull. To simplify the notation, we will generally use only indices when referring to points. We write $\{i, \ldots, j\}$ to represent the sequence $i, i+1, i+2, \ldots, j-1, j$. All operations are calculated modulo $2n$. Note that i is not necessarily less than j, and that $\{i, \ldots, j\}$ is not the same as $\{j, \ldots, i\}$.

A bichromatic set of points is a point set P equipped with a coloring function $\mathfrak{c} : P \to \{0, 1\}$ that classifies each point as either "red" (points for which $\mathfrak{c}(p) = 0$) or "blue" (points for which $\mathfrak{c}(p) = 1$). We usually denote these sets by R and B respectively, with $P = R \cup B$ and $|R| = |B| = n$, and again, we assume $n > 1$.

We say that two line segments s and t in the plane *cross* if there is a point on the plane which is not an endpoint of either s and t that belongs to both s and t. In particular, note that if $s = t$, then s and t cross each other.

The *convex hull* of a point set is the smallest convex polygon that contains all the points of it. The *convex layers* or the *onion decomposition* of a set of points are a sequence of nested convex polygons having the points as their vertices. The outermost one is the convex hull of the points and the rest are formed in the same way recursively. The innermost layer may be degenerate, consisting only of one or two points. The number of polygons in onion decomposition of a point set is called its *layer depth*.

A *perfect matching* on the set P is a set of n straight line segments whose endpoints are points in P such that each point is the endpoint of exactly one line segment. For bichromatic points sets, we further require that each line segment has one red and one blue endpoint. If the line segments do not cross, we refer to such a matching as a (bichromatic) non-crossing matching. All matchings are both perfect and non-crossing unless mentioned otherwise.

We usually use the notation M or N to refer to matchings. With a slight abuse of notation, given a matching M over P and a point $p \in P$, we use $M(p)$ to denote the matched partner of p in M, that is, the point q such that the segment connecting p and q belongs to M. Two matchings M and N are called *disjoint* if the matched partners of all points in p are different in both, i.e, for all $p \in P$, we have that $M(p) \neq N(p)$, and *compatible* if the segments in the multiset $M \cup N$ do not cross. Note that all compatible matchings are disjoint, while the converse may not be true.

We define the *distance* between two matchings M and N over a point set P, denoted $D_P(M, N)$, as $\min_{p \in P}(\text{dist}(M(p), N(p)))$, where $dist(\cdot, \cdot)$ denotes the Euclidean distance between two points. We also refer to this as the diversity of the set $\{M, N\}$ or the diversity between M and N. Further, we say that a pair of matchings M and N over P are k-diverse if $D_P(M, N) \geq k$. Note that if M and N are *not* disjoint, then they are 0-diverse. If the point set P is clear from the context, we may drop the subscript P from the notation for distances and diversity.

We now introduce some terminology that is relevant to bichromatic point sets.

Definition 1 (Balanced, Blue-heavy, Red-heavy) *A set of points is balanced if it contains the same number of red and blue points. If the set has more red (blue) points than blue (red), we say that it is red-heavy (blue-heavy).*

Lemma 1 (Savić and Stojaković (2022)) *Every balanced set of points can be matched.*

Definition 2 (Feasible pair.) *We say that (i, j) is a feasible pair if there exists a matching containing (i, j). We refer to i as a feasible neighbour of j and vice versa.*

Lemma 2 (Savić and Stojaković (2022)) *A pair (i, j) is feasible if and only if i and j have different colors and $\{i, \ldots, j\}$ is balanced.*

Definition 3 (Functions o^+ and o^-.) *[Savić and Stojaković (2022)] By $o^+(i)$ we denote the first point starting from i in the positive direction such that $(i, o^+(i))$ is feasible. By $o^-(i)$ we denote the first point starting from i in the negative direction such that $(o^-(i), i)$ is feasible.*

As we assume that the given point set is balanced, Lemma 2 guarantees that both o^+ and o^- are well-defined. It also turns out that o^- is the inverse function of o^+ as mentioned in Savić and Stojaković (2022). We denote the composition of o^+ function k times on a point p as $o^k(p)$ and also use the notation $o(p)$ to mean $o^+(p)$.

Definition 4 (Orbit) *[Savić and Stojaković (2022)]*

An orbit of i, denoted by $\mathcal{O}(i)$, is defined by $\mathcal{O}(i) := \left\{o^k(i) : k \in \mathbb{Z}\right\}$. By $\mathcal{O}(P)$ we denote the set of all orbits of a convex point set P, that is $\mathcal{O}(P) := \{\mathcal{O}(i) : i \in P\}$.

3 DC-NCM for points in Convex Position

Suppose that the points of P are in convex position. Let $\mathcal{F}$ be a collection of even-length simple convex polygons, each of length ≥ 4. We say that $\mathcal{F}$ is a *feasible collection of polygons on* P if the following hold true:

- Every $p \in P$ is a vertex of exactly one polygon in $\mathcal{F}$, and every polygon in $\mathcal{F}$ has all its vertices in P.
- No edge of a polygon in $\mathcal{F}$ crosses an edge of another polygon in $\mathcal{F}$.

For any even-length simple polygon T of length ≥ 4,

- let $partners(T)$ denote the set of all unordered pairs $\{u, v\}$ of vertices of T such that exactly one vertex of T appears between u and v, when one traverses from u to v in counter-clockwise direction along the boundary of T.
- let $quality(T)$ denote the minimum of $\text{dist}(u, v)$ over all pairs $\{u, v\}$ in $partners(T)$.

For any feasible collection $\mathcal{F}$ of polygons on P, let $quality(\mathcal{F})$ denote the minimum of $quality(T)$ over all polygons T in $\mathcal{F}$.

Note that for any $k > 0$, the following are equivalent:

- There exists a pair of compatible perfect NCMs M and N on P such that $D_P(M, N) \geq k$.
- There exists a feasible collection $\mathcal{F}$ of polygons on P such that $quality(\mathcal{F}) \geq k$.

This claim follows from the fact that the union of the line segments in any pair of compatible NCMs over P is a collection of even-length simple convex polygons whose vertices partition P and do not cross, i.e, a feasible collection of polygons on P.

Thus, our goal is to find a collection $\mathcal{F}$ of feasible polygons on P for which $quality(\mathcal{F})$ is maximized.

Let A and B be non-empty sets of real numbers. We say that A *dominates* B if for every $x \in A$, we have $x \geq y$ for some $y \in B$. Note that

- If $A \subseteq B$, then A dominates B.
- A dominates B if and only if $\min(A) \geq \min(B)$.

Our dynamic programming algorithm relies on the following structural lemma, which says the following: if $\mathcal{F}$ is a feasible collection of polygons on P with quality s, then we can find a (potentially different) $\mathcal{F}'$ which is a feasible collection of polygons on P whose quality is no worse than s, and further, every polygon in $\mathcal{F}'$ has four or six vertices. This allows us to devise a polynomial time algorithm based on "guessing" the nature of the polygons that the points belong to in some final solution.

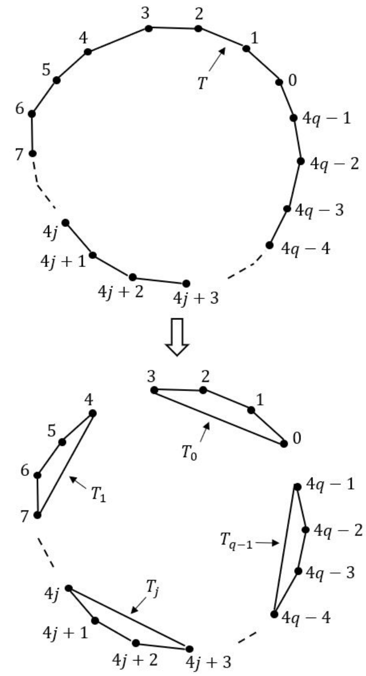


Figure 1: Breaking up a polygon on $4q$ points.

Our proof for the structural lemma considers two scenarios. First, when the number of vertices of a polygon T in $\mathcal{F}$ is a multiple of four, we simply "break" it into four-length polygons. In this situation, we introduce no new pairs into the set of matched partners (i.e, $partners(T') \subseteq partners(T)$ for any T' generated by the breaking procedure) , and so the quality of the solution is not affected. The other situation is that the number of vertices of a polygon T in $\mathcal{F}$ is of the form $4q + 2$. In this case, we find a six-length polygon T' and organize the remaining $4q - 4$ points into $(q - 1)$ polygons as in the previous case. The choice of T' is made carefully so as to ensure that the overall quality of the solution thus obtained is no worse than the original.

Lemma 3 *Let $\mathcal{F}$ be a feasible collection of polygons on P. Then, there exists a feasible collection $\mathcal{F}'$ of polygons on P such that*

- *Every polygon in $\mathcal{F}'$ has length either 4 or 6.*
- *$quality(\mathcal{F}') \geq quality(\mathcal{F})$*

Proof.
Let $\mathcal{F}'$ be a family of polygons on P obtained from $\mathcal{F}$ as follows: For each polygon T in $\mathcal{F}$ of length > 6,

Case 1: T has length $4q$, for some integer $q \geq 2$

Let $0, 1, 2 \ldots, 4q - 1$ denote the vertices of T, appearing in that order as one traverses in counter-clockwise direction along its boundary. We replace T with q simple convex polygons $T_0, T_1, \ldots, T_{q-1}$, each of length 4, where

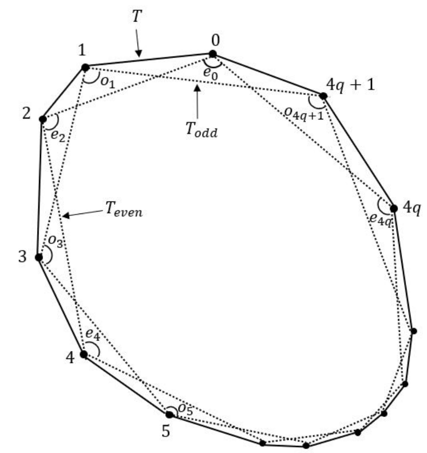


Figure 2: The even and odd polygons T_{even} and T_{odd}.

- T_0 has vertices $0, 1, 2, 3$.
- T_1 has vertices $4, 5, 6, 7$.

$\vdots$

$\vdots$

- T_{q-1} has vertices $4q-4, 4q-3, 4q-2, 4q-1$.

Let $0 \leq j \leq q-1$. Note that

$$\begin{aligned} partners(T_j) &= \Big\{\{4j, 4j+2\}, \{4j+1, 4j+3\}\Big\} \\ &\subseteq partners(T) \end{aligned}$$

So, we have

$$\min_{\{u,v\} \in partners(T_j)} \Big(\text{dist}(u, v)\Big) \geq \min_{\{u,v\} \in partners(T)} \Big(\text{dist}(u, v)\Big)$$

That is, $quality(T_j) \geq quality(T)$.

Thus,

$$\min_{0 \leq j \leq q-1} \Big(quality(T_j)\Big) \geq quality(T)$$

Case 2: T has length $4q+2$, for some integer $q \geq 2$

Let $0, 1, 2, \ldots, 4q+1$ denote the vertices of T, appearing in that order as one traverses in counter-clockwise direction along its boundary.

Let T_{even} and T_{odd} denote the simple convex polygons, each of length $2q+1$, on the vertices $0, 2, 4, \ldots, 4q$ and $1, 3, 5, \ldots, 4q+1$ respectively. Let $e_0, e_2, e_4, \ldots, e_{4q}$ denote the interior angles of T_{even}, at the vertices $0, 2, 4, \ldots, 4q$ respectively. Let $o_1, o_3, o_5, \ldots, o_{4q+1}$ denote the interior angles of T_{odd}, at the vertices $1, 3, 5, \ldots, 4q+1$ respectively.

For any simple convex polygon, since its exterior angles sum up to 2π, at most two of them are $> \frac{2\pi}{3}$. So, at most two of its interior angles are $< \frac{\pi}{3}$.

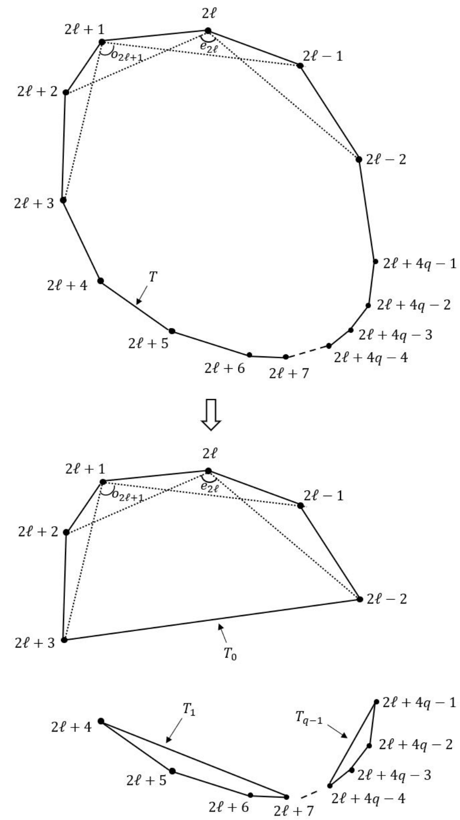


Figure 3: Breaking up a polygon on $4q+2$ points.

Thus, each of T_{even} and T_{odd} has at most two interior angles that are $< \frac{\pi}{3}$. That is, at most two of $e_0, e_2, \ldots, e_{4q}$ are $< \frac{\pi}{3}$, and at most two of $o_1, o_3, \ldots, o_{4q+1}$ are $< \frac{\pi}{3}$. So, among the ≥ 5 pairs of angles $(e_0, o_1), (e_2, o_3), (e_4, o_5), \ldots, (e_{4q}, o_{4q+1})$, there is at least one pair, say $(e_{2\ell}, o_{2\ell+1})$, such that each of $e_{2\ell}$ and $o_{2\ell+1}$ is $\geq \frac{\pi}{3}$.

We replace T with a simple convex polygon T_0 of length 6, and $q-1$ simple convex polygons $T_1, \ldots, T_{q-1}$, each of length 4 (c.f. Figure 3), where:

- T_0 has vertices $2\ell-2, 2\ell-1, 2\ell, 2\ell+1, 2\ell+2, 2\ell+3$.
- T_1 has vertices $2\ell+4, 2\ell+5, 2\ell+6, 2\ell+7$.
- T_2 has vertices $2\ell+8, 2\ell+9, 2\ell+10, 2\ell+11$.

$\vdots$

$\vdots$

- T_{q-1} has vertices $2\ell+4q-4, 2\ell+4q-3, 2\ell+4q-2, 2\ell+4q-1$.

Here, the additions are modulo $4q+2$.

Let $1 \leq j \leq q-1$. Note that

$$partners(T_j) = \Big\{\{2\ell+4j, 2\ell+4j+2\},\ \{2\ell+4j+1, 2\ell+4j+3\}\Big\} \subseteq partners(T)$$

So, we have

$$\min_{\{u,v\}\in partners(T_j)}\Big(\mathrm{dist}(u,v)\Big) \geq \min_{\{u,v\}\in partners(T)}\Big(\mathrm{dist}(u,v)\Big)$$

That is, $quality(T_j) \geq quality(T)$.

Next, we show that $quality(T_0) \geq quality(T)$.

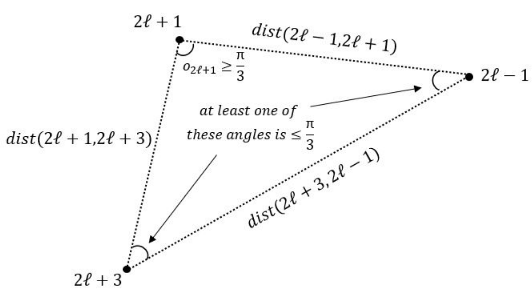
Let

$$A := \Big\{\mathrm{dist}(u,v) \mid \{u,v\} \in partners(T_0)\Big\}$$

$$B := \Big\{\mathrm{dist}(u,v) \mid \{u,v\} \in partners(T)\Big\}$$

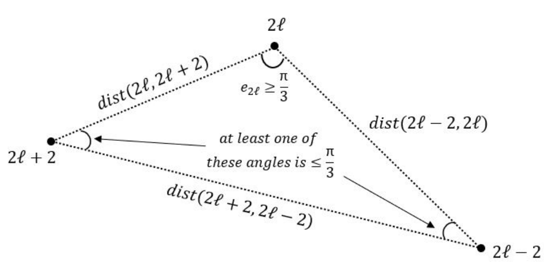
Note that

$$partners(T_0) \setminus partners(T) = \Big\{\{2\ell+3, 2\ell-1\},\ \{2\ell+2, 2\ell-2\}\Big\}$$



Consider the triangle formed by the points $2\ell-1$, $2\ell+1, 2\ell+3$. Here, as $o_{2\ell+1} \geq \frac{\pi}{3}$,

$$\mathrm{dist}(2\ell+3, 2\ell-1) \geq \min\begin{pmatrix}\mathrm{dist}(2\ell-1, 2\ell+1), \\ \mathrm{dist}(2\ell+1, 2\ell+3)\end{pmatrix}.$$



Consider the triangle formed by the points $2\ell-2$, $2\ell, 2\ell+2$. Here, as $e_{2\ell} \geq \frac{\pi}{3}$,

$$\mathrm{dist}(2\ell+2, 2\ell-2) \geq \min\begin{pmatrix}\mathrm{dist}(2\ell-2, 2\ell), \\ \mathrm{dist}(2\ell, 2\ell+2)\end{pmatrix}.$$

Also, note that $partner(T)$ contains the pairs $\{2\ell-1, 2\ell+1\}, \{2\ell+1, 2\ell+3\}, \{2\ell-2, 2\ell\}, \{2\ell, 2\ell+2\}$. Therefore, A dominates B and so, $\min(A) \geq \min(B)$.

That is,

$$\min_{\{u,v\}\in partners(T_0)}\Big(\mathrm{dist}(u,v)\Big) \geq \min_{\{u,v\}\in partners(T)}\Big(\mathrm{dist}(u,v)\Big)$$

Thus, $quality(T_0) \geq quality(T)$.

Hence, we have

$$\min_{0\leq j\leq q-1}\Big(quality(T_j)\Big) \geq quality(T),$$

and this concludes the proof. $\square$

Based on the lemma, we have the following dynamic programming approach: Let $0, 1, \ldots, 2n-1$ denote the points of P in counter-clockwise order. For every $0 \leq i, j \leq 2n-1$ such that $(j-i)$ is odd, let $Q_{i,j}$ denote the set of points $\{i, i+1, \ldots, j\}$ and let:

$$\mathbb{T}(i,j) = \begin{cases} \max D_{Q_{ij}}(M,N) & \text{if } j-i \geq 3, \\ -\infty & \text{if } j-i = 1, \\ +\infty & \text{if } j-i < 0; \end{cases}$$

where the max is taken over all pairs of disjoint compatible perfect NCMs M and N over the point set $Q_{i,j}$.

Note that $\mathbb{T}(0, 2n-1)$ is the value of the optimal solution. We compute and store $\mathbb{T}(i,j)$'s using the following recurrence:

$$\mathbb{T}(i,j) = \begin{cases} \max(\alpha(i,j), \beta(i,j)) & \text{if } j-i \geq 5, \\ \alpha(i,j) & \text{if } j-i = 3, \\ -\infty & \text{if } j-i = 1, \\ +\infty & \text{if } j-i < 0, \end{cases}$$

where $\alpha(i,j)$ is given by:

$$\max_{\substack{i < p_1 < p_2 < p_3 \leqslant j: \\ p_1 - i \text{ is odd} \\ p_2 - p_1 \text{ is odd} \\ p_3 - p_2 \text{ is odd}}} \quad \min\begin{pmatrix} \mathrm{dist}(i, p_2), \\ \mathrm{dist}(p_1, p_3), \\ \mathbb{T}(i+1, p_1-1), \\ \mathbb{T}(p_1+1, p_2-1), \\ \mathbb{T}(p_2+1, p_3-1), \\ \mathbb{T}(p_3+1, j) \end{pmatrix}$$

and $\beta(i,j)$ is given by:

$$\max_{\substack{i<q_1<q_2<q_3<q_4<q_5\leqslant j:\\ q_1-i \text{ is odd}\\ q_2-q_1 \text{ is odd}\\ q_3-q_2 \text{ is odd}\\ q_4-q_3 \text{ is odd}\\ q_5-q_4 \text{ is odd}}} \min \begin{pmatrix} \operatorname{dist}(i,q_2), \\ \operatorname{dist}(q_1,q_3), \\ \operatorname{dist}(q_2,q_4), \\ \operatorname{dist}(q_3,q_5), \\ \operatorname{dist}(q_4,i), \\ \operatorname{dist}(q_5,q_1), \\ \mathbb{T}(i+1,q_1-1), \\ \mathbb{T}(q_1+1,q_2-1), \\ \mathbb{T}(q_2+1,q_3-1), \\ \mathbb{T}(q_3+1,q_4-1), \\ \mathbb{T}(q_4+1,q_5-1), \\ \mathbb{T}(q_5+1,j) \end{pmatrix}.$$

We remark that the recurrences are well-defined. The overall intuition for the recurrences above is the following: fix an arbitrary solution that has the property guaranteed by Lemma 3. We attempt to "guess" the type and vertices of the polygon that the first point belongs to in this solution. For each fixed guess, we have a natural partition of the remaining points into smaller sub-instances (see Figures 4 and 5). It is easy to identify invalid guesses, by which we mean a polygon which is such that there is no solution that contains it.

For any valid guess, the recurrence gives us the best possible extension, i.e, the best possible diversity achievable among solutions that contain the guessed polygon. All that remains is to pick the best choice among all choices of polygons that contain the first point. The overall running time is polynomially bounded because we only have to worry about polygons with a constant number of vertices. We make this argument more explicit in the Appendix. We also note that the running time of our algorithm is $\mathcal{O}(n^7)$ since the DP table has $\mathcal{O}(n^2)$ indices and the computation at each index is $\mathcal{O}(n^5)$.

We now sketch the correctness of the dynamic programming approach proposed in the context of Theorem 3. Consider the subproblem given by the points $i, i+1, \ldots, j$. Consider the space of all solutions $\mathcal{S}$ that have the property guaranteed by Lemma 3 and partition it into two parts: $\mathcal{S}_4 \subseteq \mathcal{S}$ consists of all solutions where the point i belongs to a polygon with four sides; and $\mathcal{S}_6 \subseteq \mathcal{S}$ consists of all solutions where the point i belongs to a polygon with six sides.

Let $A^\star$ and $B^\star$ denote arbitrary optimal solutions among all the solutions in $\mathcal{S}_4$ and $\mathcal{S}_6$, respectively. Further, let $a^\star$ and $b^\star$ denote the corresponding costs. Note that the cost of the optimal solution for this subproblem is $\max(a^\star, b^\star)$.

We now argue that $\alpha(i,j)$ correctly computes the value of $a^\star$. Once again, for every choice of points $i < p_1 < p_2 < p_3 \leq j$ in $\binom{Q_{i+1,j}}{3}$, let $\mathcal{S}_4[[p_1,p_2,p_3]]$ denote the set of all solutions in $\mathcal{S}_4$ where the polygon containing the point i also contains the points p_1, p_2, p_3. Note that if it is *not* the case that $p_1 - i$ is odd and $p_2 - p_1$ is odd and $p_3 - p_2$ is odd , then $\mathcal{S}_4[[p_1,p_2,p_3]] = \emptyset$, since for any such combination of points, there is no valid solution containing the polygon formed by the points $\{i, p_1, p_2, p_3\}$. For any valid combination, we know that the best solution in $\mathcal{S}_4[[p_1,p_2,p_3]]$ is captured by taking the union of the best solutions for the following subinstances: $(i+1, p_1-1)$, (p_1+1, p_2-1), (p_2+1, p_3-1), and (p_3+1, j) corresponding to the four "chunks" of points "carved out" by the polygon (see Figure 4); along with the polygon formed by the points $\{i, p_1, p_2, p_3\}$. Note that there are no points *inside* the polygon whose vertices are $\{i, p_1, p_2, p_3\}$ since the original point set is in convex position. Further, note that it is reasonable to consider these subinstances independently since no solution that contains the polygon formed by $\{i, p_1, p_2, p_3\}$ will contain a polygon with points from two distinct segments among the segments listed above. The proof can now be completed using a standard strong induction argument, and we defer the details to the full version.

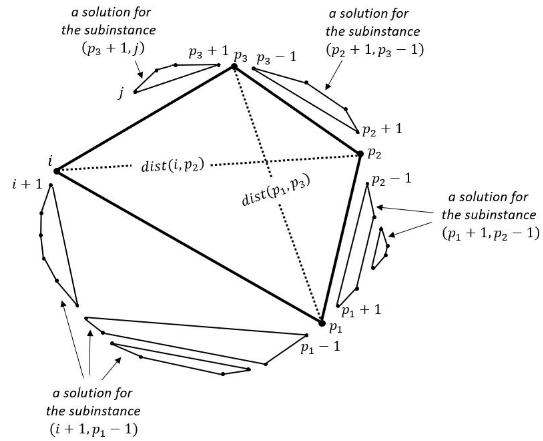


Figure 4: An example of how a base polygon divides the subproblem on $Q_{i,j}$ further into four smaller instances.

4 Concluding Remarks

We introduced the notion of diverse non-crossing matchings. While we show that DIVERSE COMPATIBLE NCMs can be solved in polynomial time for points in convex position, the complexity of the closely related problem DIVERSE NCMs (where we drop the demand for compatibility from the solution matchings) remains open even for convex point sets. The complexity of all problems considered for more general inputs remains open. We also believe that exploring other notions of diversity, based on either different aggregation func-

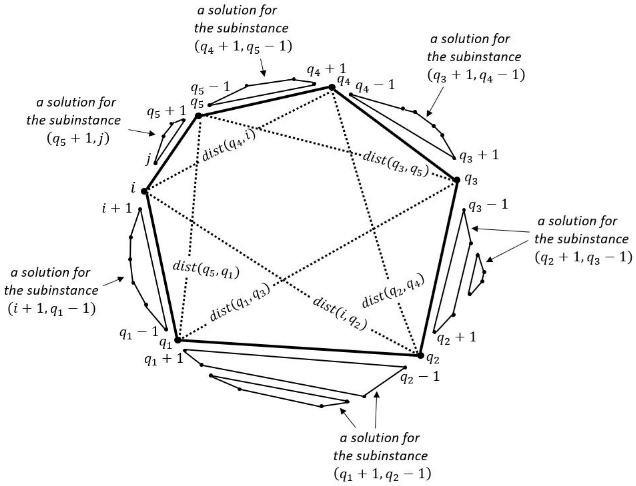


Figure 5: An example of how a base polygon divides the subproblem on $Q_{i,j}$ further into six smaller instances.

tions (e.g, sum instead of minimum), or other notions of distance (different from Euclidean), would also pose interesting directions for future research. We also propose to study the problems proposed here for more than two matchings.

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