Minimum Enclosing Spherical/Cylindrical Shells in High-Dimensional Streams

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Abstract

Given a point set P in $\mathbb{R}^d$, the minimum enclosing spherical (resp., cylindrical) shell problem is to find a sphere (resp., a cylinder) best fitting P. We show that in the single-pass streaming model, any algorithm for the minimum enclosing spherical/cylindrical shell problem with an approximation factor better than $d^{1/3}/4$ requires a memory size exponential in d.

1 Introduction

Shape fitting is a fundamental problem in computational geometry, with various applications to machine learning, data mining, statistics, and computer vision. The objective in the shape fitting problem is to find a shape from a certain family of shapes, best fitting a given point set. Examples of geometric shape fitting problems include minimum enclosing ball, width, minimum-radius enclosing cylinder, and minimum-width enclosing spherical/cylindrical shell.

In this paper, we consider shape fitting problems in the streaming model, where input points arrive one at a time, and the algorithm has a limited storage, so it cannot keep all the points received so far in memory. Moreover, we are considering the problems in high dimensions, where the dimension, d, can be arbitrarily large.

In fixed dimensions, a $(1+\varepsilon)$-approximation for many geometric shape fitting problems can be computed efficiently in linear time using the idea of core-sets [1]. The idea can be extended to the streaming model as well, leading to efficient $(1+\varepsilon)$-approximation streaming algorithms that require only $1/\varepsilon^{O(d)}$ space [1, 4, 7].

In high dimensions, however, the above $(1+\varepsilon)$-factor approximation algorithms are not applicable, due to their exponential dependency in d. Nevertheless, there are a few results on shape fitting in high-dimensional streams. For the minimum enclosing ball problem, Zarrabi-Zadeh and Chan [8] presented a simple 3/2-approximation algorithm that works in any dimension, using only $O(d)$ space. Agarwal and Sharathkumar [2] presented another $O(d)$-space streaming algorithm with an approximation factor of $\frac{1+\sqrt{3}}{2}+\varepsilon \approx 1.37$. The approximation factor of their algorithm was later improved to 1.22 by Chan and Pathak [5], getting very close to the lower bound of $\frac{1+\sqrt{2}}{2} \approx 1.207$ known for the problem [2]. For the minimum enclosing cylinder problem, Chan [4] gave an elegant $(5+\varepsilon)$-approximation streaming algorithm using only $O(d)$ space. For the width problem, Agarwal and Sharathkumar [2] proved a lower bound of $d^{1/3}/8$ on the approximation factor of any streaming algorithm for the problem whose memory size is subexponential in d.

In this paper, we study two other important shape fitting problems in high dimensions, namely the minimum enclosing spherical shell and the minimum enclosing cylindrical shell in the data stream model. We show that any (randomized) streaming algorithms that approximates the minimum enclosing spherical/cylindrical shell to within a factor better than $d^{1/3}/4$ (with probability at least 2/3) requires a memory size exponential in d.

To prove our lower bounds, we use the well-known lower bound on the one-round communication complexity of the index problem [6], in the same way used by Agarwal and Sharathkumar [2]. However, technical details of our proofs are non-trivial, and require carefully exploiting geometric properties of spherical/cylindrical shells. To the best of our knowledge, the two problems studied in this paper—despite being central in geometric shape fitting—have not been studied before in high-dimensional streams. As such, this work is an important step towards better understanding the limitations of shape fitting in high-dimensional data streams.

2 Preliminaries

Given a point $c \in \mathbb{R}^d$ and two positive real numbers r_1 and r_2 such that $r_1 < r_2$, the closed region between two concentric spheres of radii r_1 and r_2 centered at c is called a *spherical shell*. The *width* of the spherical shell is defined as $r_2 - r_1$. Given a line ℓ in $\mathbb{R}^d$ and two positive real numbers r_1 and r_2 such that $r_1 < r_2$, the closed region between two coaxial cylinders with axis ℓ and radii r_1 and r_2 is called a *cylindrical shell*. Likewise, the value $r_2 - r_1$ is called the *width* of the cylindrical shell. The *minimum enclosing spherical (resp., cylindri-*

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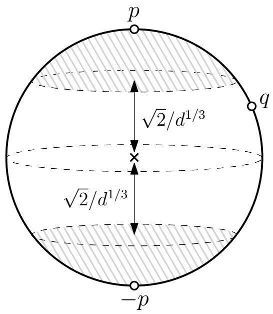

Figure 1: Two points p and q on $\mathbb{S}^{d-1}$.

cal) shell problem is to find a minimum-width spherical (resp., cylindrical) shell that encloses the whole input point set.

Let $\mathbb{S}^{d-1}$ be the a sphere in $\mathbb{R}^d$ centered at origin. Namely, $\mathbb{S}^{d-1} = \{p \in \mathbb{R}^d \mid \|op\| = 1\}$, where o denotes the origin. We call a point set P *symmetric* if for every point p in P, $-p$ is also in P. Given two points/vectors u and v in $\mathbb{R}^d$, we denote by $u \cdot v$ the inner product of u and v. We also denote by $\exp(x)$ the function e^x, where e is the natural number. The following lemma, which is stated in a slightly different form in [2], will be used to derive our lower bounds.

Lemma 1 *There is a symmetric point set $K \subseteq \mathbb{S}^{d-1}$ of size $\Omega(\exp(d^{1/3}))$ such that for every pair of distinct points $p, q \in K$, if $q \neq -p$, then $|p \cdot q| \leq \sqrt{2}/d^{1/3}$.*

Proof. Let $p \in \mathbb{S}^{d-1}$ and $0 < \delta \leq 1$. The hyperplane at distance δ from the origin and normal to p partitions $\mathbb{S}^{d-1}$ into two parts. We denote the smaller part by $\text{cap}(p, \delta)$. Now, consider the point set K returned by the following algorithm.

Algorithm 1 Well-Separated Point Set

```
F ← S^{d-1}, K ← ∅
while F ≠ ∅ do
    pick an arbitrary point p in F
    K ← K ∪ {p, −p}
    F ← F \ (cap(p, √2/d^{1/3}) ∪ cap(−p, √2/d^{1/3}))
return K
```

Let p and q be two points in K, such that $q \neq -p$. Suppose that p is added to K before q by the algorithm. Then, it is clear by our construction that q is at distance at most $\sqrt{2}/d^{1/3}$ from the hyperplane passing through the origin and normal to p, which means $|p \cdot q| \leq \sqrt{2}/d^{1/3}$. (See Figure 1.)

To analyze the size of K, let $S(X)$ denote the area of a surface X. It is well-known [3] that for any $p \in \mathbb{S}^{d-1}$ and any $0 \leq \delta \leq 1$,

$$\frac{S(\text{cap}(p, \delta))}{S(\mathbb{S}^{d-1})} \leq \exp(-d\delta^2/2).$$

Therefore, at any iteration of the algorithm, no more than $2/\exp(d^{1/3})$ of the surface of $\mathbb{S}^{d-1}$ is removed from F. Hence, $|K| = \Omega(\exp(d^{1/3}))$. $\square$

3 Spherical Shell

In this section, we provide a lower bound on the approximation factor of any streaming algorithm for the minimum enclosing spherical shell problem whose storage size is sub-exponential in d. The following lemma provides the main ingredient of our proof.

Lemma 2 *Let $\{\vec{u}_1, \ldots, \vec{u}_d\}$ be a set of orthogonal unit vectors in $\mathbb{R}^d$. Then any spherical shell that encloses the point set $P = \{o, \pm\vec{u}_1\} \cup \bigcup_{i=2}^{d} \{\pm\sqrt{d}\vec{u}_i, \pm(\sqrt{d}+2)\vec{u}_i\}$ has width at least $\frac{\sqrt{2}}{2}$.*

Proof. Suppose that the minimum spherical shell enclosing P is centered at a point $a = (a_1, \ldots, a_d)$, where $a_i = a \cdot \vec{u}_i$. Due to symmetry of P around the origin, we can assume without loss of generality that $a_i \geq 0$ for all $i \in \{1, \ldots, d\}$. Moreover, by symmetry of definition of P on dimensions 2 to d, we can assume without loss of generality that $a_2 \geq a_i$ for all $i \in \{3, \ldots, d\}$, i.e., a_2 is the biggest number among $a_2, \ldots, a_d$. We consider the following four cases:

(i) $a_1 \leq a_2\sqrt{d}$ and $a_2 \geq 1$

(ii) $a_1 \leq a_2\sqrt{d}$ and $a_2 \leq 1$

(iii) $a_1 \geq a_2\sqrt{d}$ and $a_1 \geq 1$

(iv) $a_1 \geq a_2\sqrt{d}$ and $a_1 \leq 1$

Let $w_a(P)$ denote the width of the minimum spherical shell centered at a enclosing P, i.e., $w_a(P) = \max_{p \in P} \|pa\| - \min_{p \in P} \|pa\|$. To prove the lemma, it is enough to show that in all the above four cases, $w_a(P) \geq \frac{\sqrt{2}}{2}$. We will prove the first case here, and leave the rest to the appendix. By the definition of $w_a(P)$ we have

$$\begin{aligned} w_a(P) &\geq w_a\Big(\{\pm\sqrt{d}\vec{u}_2\}\Big) \\ &= \sqrt{a_1^2 + (a_2 + \sqrt{d})^2 + a_3^2 + \cdots + a_d^2} \\ &\quad -\sqrt{a_1^2 + (a_2 - \sqrt{d})^2 + a_3^2 + \cdots + a_d^2} \\ &= \sqrt{\Sigma_{i=1}^d a_i^2 + d + 2a_2\sqrt{d}} \\ &\quad -\sqrt{\Sigma_{i=1}^d a_i^2 - d + 2a_2\sqrt{d}}. \end{aligned}$$

Note that the function $f(x) = \sqrt{x + c} - \sqrt{x}$ is decreasing in x, for all positive c. Therefore, in the above expression, if we increase both values under the roots by the same amount, the expression becomes smaller. Therefore,

$$\begin{aligned}
w_a(P) &\geq \sqrt{2da_2^2 + d + 4a_2\sqrt{d}} - \sqrt{2da_2^2 + d} \\
&\quad (\text{as } a_2\sqrt{d} \geq a_1 \text{ and } a_2 \geq a_i \text{ for } i = 3, \ldots, d) \\
&\geq \sqrt{3da_2^2 + 4a_2\sqrt{d}} - \sqrt{3da_2^2} \qquad (\text{as } a_2 \geq 1) \\
&\geq \sqrt{3}\left(\sqrt{da_2^2 + a_2\sqrt{d} + \frac{a_2\sqrt{d}}{3}} - \sqrt{da_2^2}\right) \\
&\geq \sqrt{3}\left(\sqrt{da_2^2 + a_2\sqrt{d} + \frac{1}{4}} - \sqrt{da_2^2}\right) \\
&\quad (\text{as } a_2, d \geq 1 \text{ and } a_2\sqrt{d}/3 > 1/4) \\
&= \sqrt{3}\left(\sqrt{\left(a_2\sqrt{d} + \frac{1}{2}\right)^2} - \sqrt{da_2^2}\right) \\
&= \frac{\sqrt{3}}{2},
\end{aligned}$$

which implies $w_a(P) \geq \frac{\sqrt{2}}{2}$ in Case (i). The proof of the other three cases are provided in the appendix. □

Now, we are ready to provide our lower bound. Suppose $\mathbb{A}$ is a streaming algorithm that approximates the minimum enclosing spherical shell within a factor better than $d^{1/3}/4$ with probability at least $2/3$. We reduce from the following problem in communication complexity:

Index Problem. *Alice has a binary string $a_1 \ldots a_k$ and Bob has an index $i \in \{1, \ldots, k\}$. Alice can send Bob a message to inform him about her string and then Bob should determine whether a_i is 0 or 1.*

It is known [6] that in any algorithm for the index problem that succeeds with probability at least $2/3$, the size of the message sent by Alice to Bob is $\Omega(k)$. By reducing from the index problem to the minimum enclosing spherical shell problem, we will show that the space required by $\mathbb{A}$ is $\Omega(\exp(d^{1/3}))$.

Let d be the smallest integer such that $k < \exp(d^{1/3})$. By Lemma 1, it is possible to choose a set K of k pair of well-separated points on $\mathbb{S}^{d-1}$. Let K^+ be the subset of points of K lying on the hemisphere $x_1 \geq 0$, and let $f : \{1, \ldots, k\} \to K^+$ be a one-to-one function. We assume that Alice and Bob are both aware of f and K, as these are independent of Alice's string or Bob's index. Alice gives the points $\{\pm f(j) \,|\, a_j = 1\}$ to $\mathbb{A}$ and then sends the working space of $\mathbb{A}$ to Bob. Bob then gives the set of points $\{o\} \cup \bigcup_{j=2}^{d}\{\pm\sqrt{d}\vec{u}_j, \pm(\sqrt{d}+2)\vec{u}_j\}$ to $\mathbb{A}$, where o is the origin and $\{\vec{u}_2, \ldots, \vec{u}_d\}$ is a set of orthogonal unit vectors which are all orthogonal to the point $f(i)$. If the output of $\mathbb{A}$ is a spherical shell of width less than $\frac{\sqrt{2}}{2}$, then Bob claims that $a_i = 0$, otherwise, he claims $a_i = 1$. We show that with probability at least $2/3$ he is true in his claim.

Suppose $a_i = 0$. In this case, all of the points that Alice and Bob have given to $\mathbb{A}$ are at most at distance $\sqrt{2}/d^{1/3}$ from the hyperplane passing through the origin and normal to the point $f(i)$. Therefore, a spherical shell centered at a point along $f(i)$ lying at infinity encloses all of the points with a width of at most $2\sqrt{2}/d^{1/3}$. Hence, due to the approximation factor of $\mathbb{A}$, the output will be a spherical shell of width less than $\frac{\sqrt{2}}{2}$, with probability at least $2/3$.

On the other hand, if $a_i = 1$, due to Lemma 2, there is no spherical shell of width less than $\frac{\sqrt{2}}{2}$ that encloses Bob's points plus the pair of points $\pm f(i)$. Thus, the output that Bob receives has width at least $\frac{\sqrt{2}}{2}$.

Therefore, the protocol presented for the index problem works with probability at least $2/3$. Hence, the size of the working space that Alice sends to Bob is $\Omega(k) = \Omega(\exp(d^{1/3}))$. We can conclude the following theorem.

Theorem 3 *Given a set P of n points in $\mathbb{R}^d$, any streaming algorithm that approximates the minimum enclosing spherical shell of P to within a factor better than $d^{1/3}/4$ with probability at least $2/3$ requires $\Omega(\min\{n, \exp(d^{1/3})\})$ space.*

4 Cylindrical Shell

In this section, we prove a lower bound on the approximation factor of any streaming algorithm for the minimum enclosing cylindrical shell problem whose memory size is sub-exponential in d. Again, we start with the following crucial lemma.

Lemma 4 *Let $\{\vec{u}_1, \ldots, \vec{u}_d\}$ be a set of orthogonal unit vectors in $\mathbb{R}^d$. Then any cylindrical shell that encloses the point set $P = \{o, \pm\vec{u}_1\} \cup \bigcup_{i=2}^{d}\{\pm\sqrt{d}\vec{u}_i\}$ has width at least $\frac{\sqrt{2}}{2}$.*

Proof. Let ℓ be a line in $\mathbb{R}^d$, α be a point on ℓ, and $\vec{n}$ be a unit vector in the direction of ℓ. Then, any point on ℓ can be represented by $\alpha + t\vec{n}$, where t is a real number. Note that such a presentation is not unique. In particular, we can choose the point α such that α is orthogonal to $\vec{n}$. This condition does not lose generality, as every point $\alpha + t\vec{n}$ can be rewritten as $\alpha' + t'\vec{n}$, where $\alpha' = \alpha - \frac{\alpha\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$, $t' = t + \frac{\alpha\cdot\vec{n}}{\vec{n}\cdot\vec{n}}$, and α' is orthogonal to $\vec{n}$. Henceforth, we assume that the standard form of presenting a line has the above condition.

Now, let $\ell = \{\alpha + t\vec{n}\}$, where α is orthogonal to $\vec{n}$. We write α and $\vec{n}$ as $(a_1, \ldots, a_d)$ and $(n_1, \ldots, n_d)$, respectively, where $a_i = \alpha \cdot \vec{u}_i$ and $n_i = \vec{n} \cdot \vec{u}_i$. The perpendicularity condition ensures that $\sum_{i=1}^{d} a_i n_i = 0$. Let $\operatorname{dist}(p, \ell)$ denote the distance of a point p to the line ℓ.

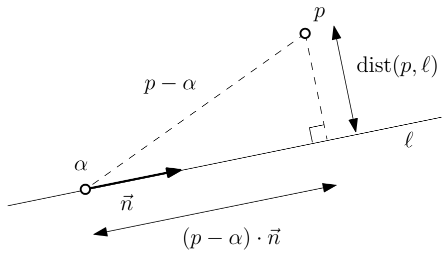

Figure 2: Computing $\mathrm{dist}(p,\ell)$.

Likewise, we write p as $(p_1,\ldots,p_d)$, where $p_i = p\cdot\vec{u}_i$. By the Pythagorean theorem,

$$\begin{aligned}\mathrm{dist}^2(p,\ell) &= |p-\alpha|^2 - ((p-\alpha)\cdot\vec{n})^2 \qquad \text{(see Fig. 2)}\\ &= \sum_{i=1}^{d}(p_i-a_i)^2 - \left(\sum_{i=1}^{d}p_in_i - \sum_{i=1}^{d}a_in_i\right)^2\\ &= \sum_{i=1}^{d}(p_i-a_i)^2 - \left(\sum_{i=1}^{d}p_in_i\right)^2\end{aligned}$$

In particular, when $p = p_k\vec{u}_k$, we have:

$$\mathrm{dist}^2(p,\ell) = \sum_{i=1}^{d}a_i^2 + p_k^2 - 2a_kp_k - p_k^2n_k^2 \tag{1}$$

Note that the minimum width of a cylindrical shell with axis ℓ enclosing point set P is equal to $\max_{p\in P}\mathrm{dist}(p,\ell) - \min_{p\in P}\mathrm{dist}(p,\ell)$. We denote this value by $w_\ell(P)$. Suppose that ℓ is the axis of a minimum-width cylindrical shell enclosing P. We can assume without loss of generality that $a_i \ge 0$ for all $i = 1,\ldots,d$, and $a_2 \ge a_i$ for all $i = 3,\ldots,d$, due to symmetry of P. We consider two cases: (i) $a_1 \ge a_2\sqrt{d}$, and (ii) $a_1 \le a_2\sqrt{d}$.

To prove the first case, we first show that $n_1^2 \le \frac{1}{2}$. Since $\sum_{i=1}^{d}a_in_i = 0$, we have

$$a_1|n_1| \;=\; \left|\sum_{i=2}^{d}a_in_i\right| \;\le\; \sum_{i=2}^{d}a_i|n_i|.$$

Therefore,

$$|n_1| \;\le\; \sum_{i=2}^{d}\frac{a_i}{a_1}|n_i| \;\le\; \frac{1}{\sqrt{d}}\sum_{i=2}^{d}|n_i|.$$

By the Cauchy-Schwarz inequality, for any two vectors u and v, $|u\cdot v|^2 \le (u\cdot u)(v\cdot v)$. As such,

$$n_1^2 \;\le\; \frac{1}{d}\left(\sum_{i=2}^{d}|n_i|\right)^2 \;\le\; \frac{d-1}{d}\sum_{i=2}^{d}n_i^2.$$

Thus,

$$n_1^2 + \frac{d-1}{d}n_1^2 \;\le\; \frac{d-1}{d}\sum_{i=1}^{d}n_i^2 \;=\; \frac{d-1}{d},$$

which implies $n_1^2 \le \frac{d-1}{2d-1} \le \frac{1}{2}$. Now,

$$\begin{aligned}w_\ell(P) &\ge \mathrm{dist}(-\vec{u}_1,\ell) - \mathrm{dist}(o,\ell)\\ &= \sqrt{\Sigma_{i=1}^{d}a_i^2 + 2a_1 + (1-n_1^2)} \qquad \text{(by (1))}\\ &\quad -\sqrt{\Sigma_{i=1}^{d}a_i^2}\\ &\ge \sqrt{\Sigma_{i=1}^{d}a_i^2 + 2a_1 + \frac{1}{2}} - \sqrt{\Sigma_{i=1}^{d}a_i^2}\\ &\ge \sqrt{2a_1^2 + 2a_1 + \frac{1}{2}} - \sqrt{2a_1^2}\\ &\quad (\text{as } a_1 \ge a_i\sqrt{d}, \text{ for } i = 2,\ldots,d)\\ &= \sqrt{2}\left(\sqrt{a_1^2 + a_1 + \frac{1}{4}} - a_1\right)\\ &= \sqrt{2}\left(\sqrt{\left(a_1+\frac{1}{2}\right)^2} - a_1\right)\\ &= \frac{\sqrt{2}}{2},\end{aligned}$$

which completes the proof of Case (i). The second case is proved in the appendix. $\square$

Now, suppose $\mathbb{A}$ is a streaming algorithm that approximates the minimum enclosing cylindrical shell to within a factor better than $d^{1/3}/4$ with probability at least $2/3$. Like the previous section, we use a reduction from the index problem. Alice gives the points $\{\pm f(j)\,|\,a_j = 1\}$ to $\mathbb{A}$, and sends the working space to Bob. Bob then gives the point set $\{o\}\cup\bigcup_{j=2}^{d}\{\pm\sqrt{d}\vec{u}_j\}$ to $\mathbb{A}$, where $\{\vec{u}_2,\ldots,\vec{u}_d\}$ is a set of orthogonal unit vectors which are all orthogonal to $f(i)$.

Let H be the hyperplane passing through the origin and normal to $f(i)$. If $a_i = 0$, all of the points given to $\mathbb{A}$ are at most at distance $\sqrt{2}/d^{1/3}$ from H. In this case, a cylindrical shell with an axis parallel to H infinitely far from H encloses all the input points with width $2\sqrt{2}/d^{1/3}$. Thus, according to the approximation factor of $\mathbb{A}$, the output of $\mathbb{A}$ is a cylindrical shell of width less than $\frac{\sqrt{2}}{2}$. On the other hand, if $a_i = 1$, then by Lemma 4, there is no cylindrical shell of width less than $\frac{\sqrt{2}}{2}$ that encloses Bob's points plus the pair of points $\pm f(i)$.

Therefore, Bob can check the output shell, and report $a_i = 0$, if the width is less than $\frac{\sqrt{2}}{2}$, and report $a_i = 1$, otherwise. This solves the index problem with probability at least $2/3$. Therefore, the working space of $\mathbb{A}$ must be $\Omega(k) = \Omega(\exp(d^{1/3}))$.

Theorem 5 *Given a set P of n points in $\mathbb{R}^d$, any streaming algorithm that approximates the minimum enclosing cylindrical shell of P to within a factor better than $d^{1/3}/4$ with probability at least $2/3$ requires $\Omega(\min\{n,\exp(d^{1/3})\})$ space.*

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Appendix

Proof of Lemma 2

Case (ii).

$$\begin{aligned}
w_a(P) &\geq w_a\Big(\{o, (\sqrt{d}+2)\vec{u}_2\}\Big)\\
&= \sqrt{a_1^2 + ((\sqrt{d}+2)-a_2)^2 + a_3^2 + \cdots + a_d^2}\\
&\quad -\sqrt{\Sigma_{i=1}^d a_i^2}\\
&\geq \sqrt{a_1^2 + (\sqrt{d}+a_2)^2 + a_3^2 + \cdots + a_d^2}\\
&\quad -\sqrt{\Sigma_{i=1}^d a_i^2}\\
&= \sqrt{\Sigma_{i=1}^d a_i^2 + 2\sqrt{d}a_2 + d} - \sqrt{\Sigma_{i=1}^d a_i^2}\\
&\geq \sqrt{2da_2^2 + 2\sqrt{d}a_2 + \frac{1}{2}} - \sqrt{2da_2^2}\\
&= \frac{\sqrt{2}}{2}.
\end{aligned}$$

Case (iii).

$$\begin{aligned}
w_a(P) &\geq w_a(\{\pm\vec{u}_1\})\\
&= \sqrt{(a_1-(-1))^2 + a_2^2 + \cdots + a_d^2}\\
&\quad -\sqrt{(a_1-1)^2 + a_2^2 + \cdots + a_d^2}\\
&= \sqrt{\Sigma_{i=1}^d a_i^2 + 2a_1 + 1} - \sqrt{\Sigma_{i=1}^d a_i^2 - 2a_1 + 1}\\
&\geq \sqrt{2a_1^2 + 4a_1} - \sqrt{2a_1^2}\\
&\geq \sqrt{2}(\sqrt{a_1^2 + \sqrt{2}a_1 + \frac{1}{2}} - a_1)\\
&= 1\\
&> \frac{\sqrt{2}}{2}.
\end{aligned}$$

Case (iv).

$$\begin{aligned}
w_a(P) &\geq w_a\Big(\{o, -\sqrt{d}\vec{u}_2\}\Big)\\
&= \sqrt{a_1^2 + (a_2-(-\sqrt{d}))^2 + a_3^2 + \cdots + a_d^2}\\
&\quad -\sqrt{\Sigma_{i=1}^d a_i^2}\\
&= \sqrt{\Sigma_{i=1}^d a_i^2 + 2\sqrt{d}a_2 + d} - \sqrt{\Sigma_{i=1}^d a_i^2}\\
&\geq \sqrt{2a_1^2 + d} - \sqrt{2a_1^2}\\
&\geq \sqrt{2+d} - \sqrt{2},
\end{aligned}$$

which is at least $\frac{\sqrt{2}}{2}$, for all $d > 2$. □

Proof of Lemma 4

Case (ii). We first prove that $d(1-n_2^2) \geq \frac{1}{2}$. Since $\sum_{i=1}^d a_i n_i = 0$, we have

$$a_2|n_2| \;=\; \left|a_1 n_1 + \sum_{i=3}^d a_i n_i\right| \;\leq\; a_1|n_1| + \sum_{i=3}^d a_i|n_i|.$$

Therefore,

$$|n_2| \;\leq\; \frac{a_1}{a_2}|n_1| + \sum_{i=3}^d \frac{a_i}{a_2}|n_i| \;\leq\; \sqrt{d}|n_1| + \sum_{i=3}^d |n_i|,$$

and hence,

$$\begin{aligned}
n_2^2 &\leq dn_1^2 + \sum_{i=3}^d 2\sqrt{d}|n_1||n_i| + \left(\sum_{i=3}^d |n_i|\right)^2\\
&\leq dn_1^2 + \sum_{i=3}^d \left(n_1^2 + dn_i^2\right) + (d-2)\sum_{i=3}^d n_i^2\\
&= (2d-2)\left(n_1^2 + \sum_{i=3}^d n_i^2\right).
\end{aligned}$$

Thus,

$$n_2^2 + (2d-2)n_2^2 \;\leq\; (2d-2)\sum_{i=1}^d n_i^2 \;=\; 2d-2.$$

Therefore, $n_2^2 \leq \frac{2d-2}{2d-1}$, and hence, $d\left(1-n_2^2\right) \geq \frac{d}{2d-1} \geq \frac{1}{2}$. We can thus conclude that

$$\begin{aligned}
w_\ell(P) &\geq \mathrm{dist}(-\sqrt{d}\vec{u}_2, \ell) - \mathrm{dist}(o, \ell)\\
&= \sqrt{\Sigma_{i=1}^d a_i^2 + 2a_2\sqrt{d} + d(1-n_1^2)} - \sqrt{\Sigma_{i=1}^d a_i^2}\\
&\geq \sqrt{2da_2^2 + 2a_2\sqrt{d} + \frac{1}{2}} - \sqrt{2da_2^2}\\
&= \frac{\sqrt{2}}{2}.
\end{aligned}$$

□