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Chapter 3.3 (Gauss-Jordan Elimination Method)

- The process used to solve a system of equations is called the **elimination method** or **Gauss-Jordan Elimination Method**
- The goal is to **reduce** the **coefficient matrix** to an **identity matrix**, so we can read the answer from the right column.
- There are **three** different operations that can be used to reduce the matrix which are called **elementary row operations**

1) Interchange two rows
2) Add a multiple of one row to another row
3) Multiply a row by a nonzero constant

Example

Solve the following linear system:

The system can be represented by the **augmented matrix:**

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 1 & 4 & 3 & 1 \\ 1 & -3 & -2 & 5 \end{array}\right]$$

Steps:

1) Get a 1 in row 1, column 1:
   Operation: Interchange row 1 and row 3

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 2 & 5 & 4 & 4 \\ 1 & -3 & -2 & 5 \end{array}\right]$$

2) Add multiples of row 1 to the other two rows (row 2 & row 3) to get zeros in the other entry of column 1:

   Operation: (2 x R1) + R2 → R2 =>

   (-1 x R1) + R3 → R3 =>

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -3 & -2 & 2 \\ 0 & -7 & -5 & 4 \end{array}\right]$$

3) Use rows below row 1 to get a 1 in row 2, column 2:
   Operation: (-1/3 x R2) → R2 =>

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -7 & -5 & 4 \end{array}\right]$$

4) Add multiples of row 2 to the other two rows (row 1 and row 3) to get zeros in the other entry of column 2:

   Operation: (-4 x R2) + R1 → R1 =>

   (7 x R2) + R3 → R3 =>

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

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5) Use rows below row 2 to get a 1 in row 3, column 3:
   Operation: (-3 x R3) → R3 =>

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

6) Add multiples of row 3 to the other two rows (row 1 and row 2) to get zeros in the other entry of column 3:

   Operation: (-1/3 x R3) + R1 → R1 =>

   (-2/3 x R3) + R2 → R2 =>

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

7) Repeat the process until it cannot be continued:
   Operation: Since all rows have been used, the matrix is in its reduced form

Initial: 2x + 5y + 4z = 4 Final: x + 0y + 0z = 3 x = 3

x + 4y + 3z = 1 0x + y + 0z = -2 => y = -2

x – 3y -2z = 5 0x + 0y + z = 2 z = 2

Chapter 3.4 (Inverse of a square matrix)

Equation:

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A^{-1} = $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:

Find the inverse of A = $\begin{bmatrix} 7 & -1 \\ -10 & 2 \end{bmatrix}$.

Solution:

$$A^{-1} = \frac{1}{14-10}\begin{bmatrix} 2 & 1 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 5/2 & 7/4 \end{bmatrix}$$

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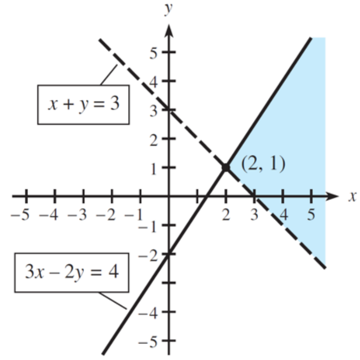
Chapter 4.1 (Linear Inequalities in two Variables)

Example:

$$\begin{cases} 3x - 2y \geq 4 \\ x + y - 3 > 0 \end{cases}$$

- Graph the following solution
- Begin by graphing the equations $3x - 2y = 4$ and $x + y = 3$ (from $x + y - 3 = 0$)

by the intercept method: Find y when $x = 0$ and find x when $y = 0$.

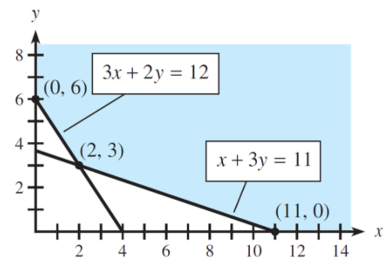


- (1) x = 0 → y = -2, y = 0 → x = ¾ ; (2) x = 0 → y = 3, y = 0 → x = 3
- We graph $3x - 2y = 4$ as a solid line and $x + y = 3$ as a dashed line
- The points that satisfy both of these inequalities lie in the intersection of the two individual solution regions
- When the two lines form a "**corner**", the points satisfy the two inequalities

Example:

Find the maximum and minimum values (if they exist) of C = x + y subject to the constraints:

3x + 2y >=12 ; x + 3y >=11;   x>=0, y>=0



- Note that the feasible region is not closed and bounded, so, we must check whether optimal values exist.

  This check is done by graphing $C = x + y$ for selected

  values of C and noting the trend.

- The corners (0, 6) and (11, 0) can be identified from the graph. The third corner, (2, 3), can be found by solving the equations

  $3x + 2y = 12$ and $x + 3y = 11$ simultaneously.

  3x + 2y = 12

(x-3) <u>-3x – 9y = -33</u>

  -7y = -21 => **y = 3**; x = 12 – 2(3) / 3 => 6 / 3=> **x = 3**

- Examining the value of C at each corner point, we have

At (0,6) C = x + y => 0 + 6 => 6 ; At (11,0) C = x + y => 11 + 0 => 11 ; At (2,3) C = x + y => 2 + 3 => 5

Minimum Value of C = x + y is 5 at (2,3) ; Maximum Value of C = x + y does not exist

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Sensitivity Analysis

- **Sensitivity Analysis** is the procedure of determining the marginal effect of input changes on the optimal solution of a model
- There are two different sensitivity analysis changes can be observed
  1) Objective Function Coefficient (OFC) 2) Constraints' right-hand side (RHS)

Example (Impact of OFC)

Original Objective Function: 7x + 5y (Profit)

Subject To: 3x + 4c <= 2400 (Carpentry Hours)

2x + y <= 1000 (Painting Hours)

y <= 450 (Max # of Chairs)

x <= 100 (Min # of tables)

x >=0, y >=0 (Nonnegativity)

What if profit contribution for tables changed from $7 (7x) to $8 (8x). How would this affect the optimal solution:

New Objective Function: **8x + 5y**

Original Optimal Point (320, 360)

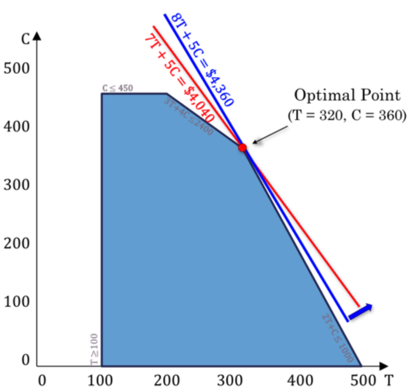
Original Value-> 7(320) + 5(360) = **4040**

Revised Optimal Point (320, 360)

Revised Value -> 8(320) + 5(360) = **4360**

There is **no effect** on the feasible region

However the slope of the **isoprofit** line changes



Impact of RHS Changes

- Impact of RHS changes depends on if the constraint is binding or nonbinding.
- **Binding** constraints pass through the optimal corner and have a **zero** slack
- **Nonbinding** constraints have a **nonzero** slack or surplus value at the optimal solution.
- **Slack** is the different between the right-hand side and left-hand side of constraint.
- **Surplus** is the different between the right-hand side and left hand side of a constraint.

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Shadow Price

- The shadow price of a constraint is the change in the optimal value of the objective function for a one-unit increase in the RHS of that constraint. (Value of an extra unit of resource)

Example

Painting hours is increased to 1300 hours:

New Profit = $4820

Old Profit = $4040

Profit Increase = **$780**

The 300 additional painting hours brings $780 in profits. Each additional painting hour will increase the profit by: $780/300 => **$2.60** → Shadow price => **$2.60**

Chapter 2.1 (Quadratic Function)

- The general equation of a **Quadratic Function is** $Y = f(x) = ax^2 + bx + c$
  Where a, b and c are real numbers and $a \neq 0$

Two methods of solving quadratic equations

1. Factoring
2. The Quadratic Formula

Example

1) Solve: $(y - 4) \times (y + 3) = 8$

Step 1: $y^2 - y - 12 = 8$

Step 2: $y^2 - y - 20 = 0$

Step 3: $(y - 5)(y + 4) = 0 \Rightarrow Y = 5;\ Y = -4$

Note:

- When solving a quadratic equation, we can use the **sign of the radicand in the quadratic formula**
- Refer $b^2 - 4ac$ as the **quadratic discriminant**
- If $b^2 - 4ac > 0$ → Equation has **two distinct real solutions**
- If $b^2 - 4ac = 0$ → Equation has exactly **one real solution**
- If $b^2 - 4ac < 0$ → Equation has **no real solutions**

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Chapter 2.2 (Quadratic Functions; Parabola)

Example

For the function **$y = 4x - x^2$,** determine whether its vertex is maximum point or a minimum point and find the coordinates of this point

The Proper form: $y = -x^2 + 4x = 0$ → a = -1 → parabola opens downward (maximum)

The vertex occurs at → x = -b/2a = -4/2(-1) = **2**

The y-coordinates of the vertex is $f(2) = -(2)^2 + 4\ (2) =$ **4** => Coordinates are (2,4)

- We can translate the parabola **vertically** to produce a new parabola that is similar to the basic parabola.
- The function **$y = x^2 + b$** has a graph which simply looks like the standard parabola with the vertex **shifted b units along the y-axis.** The vertex will be located at (0,b)
- If **b** is **positive,** then the parabola moves **upward**
- If **b** is **negative,** then the parabola moves **downward**

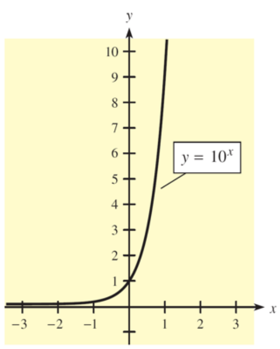
Chapter 5.1 (Exponential Functions)

- Exponential functions are a function of the form where b is a positive real number, and in which the argument x occurs as an exponent.

Example:

Graph $y = 10^x$

x	y
-3	$10^{-3} = 1/1000$
-2	$10^{-2} = 1/100$
-1	$10^{-1} = 1/10$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$



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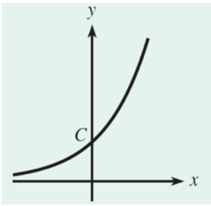
Graph of Exponential Growth Function

Function: y = f(x) = $C(a^x)$ (C > 0 , a > 1)

y-intercept: (0,C)

Graph Shape:



Domain: All real numbers; Range: y > 0

Asymptote: The x-axis (negative half)

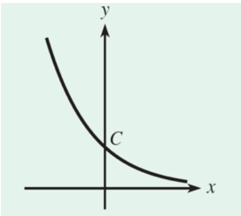
Graphs of Exponential Decay Functions

Function: y = f(x) = $C(a^{-x})$ (C > 0, a >1) or

y = f(x) = $C(b^x)$ (C > 0, 0 < b < 1)

y-intercept: (0,C)

Graph Shape:



Domain: All real numbers; Range: y > 0

Asymptote: The x-axis (positive half)

Chapter 5.2 (Logarithmic Functions and their Properties)

- **Logarithmic Functions** are the inverse of an exponential function

Logarithmic and Exponential Form Table

Logarithmic Form	Exponential Form
$\log_{10} 100 = 2$	$10^2 = 100$
$\log_{10} 0.1 = -1$	$10^{-1} = 0.1$
$\log_2 x = y$	$2^y = x$
$\log_a 1 = 0\ (a > 0)$	$a^0 = 1$
$\log_a a = 1\ (a > 0)$	$a^1 = a$

Properties of Logarithms

Property I: If a > 0, a =/ 1, then $\log_a a^x = x$, for any real number x

- To prove this result, we use the exponential form of $y = \log_a a^x$ is $a^y = a^x$, so $y = x$. This is, $\log_a a^x = x$.

Ex: (a) $\log_4 4^3 = 3$

Property II: If a > 0, a =/1, then $a^{\log_a x} = x$, for any positive real number x

Use Property II to simplify each of the following.

Ex: $2^{\log_2 4} = 4$

Property III: If a > 0, a =/ 1, and M and N are positive real numbers, then

$$\text{Log}_a(MN) = \log_a M + \log_a N$$

Ex: $\log_2 (4 \times 16) = \log_2 4 + \log_2 16 = 2 + 4 = 6$

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Property IV: If a >0, a =/ 1, and M and N are positive real numbers, then

$\text{Log}_a (M/N) = \log_a M - \log_a N$

Ex: $\text{Log}_3 (9/27) = \log_3 9 - \log_3 27 = 2 - 3 = -1$

Property V: If a > 0, a=/1, M is a positive real number and N is any real number, then

$\text{Log}_a (M^N) = N\log_a M$

Ex: $\text{Log}_3 (9^2) = 2\log_3 9 = 2 \times 2 = 4$

Chapter 7.1 (Probability; Odds)

If an event E can occur in n(E) = k ways out of n(S) = n equally likely ways, then: Pr(E) = n(E) / n(S) = k / n

Example

If a number is to be selected at random from the integers 1 through 12, what is the probability that it is divisible by 4?

Answer : The set *S* = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is an equiprobable sample space for this experiment The set *E* = {4, 8, 12} contains the numbers that are divisible by 4. Thus

$$\text{Pr (divisible by 3)} = \frac{n(E)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

If an event E is certain to occur, E contains **all of the elements of the sample space**, S.

Hence the sum of the probability weights of E is the same as that of S, so

Pr(E) = 1 if E is certain to occur

If event E is impossible, Pr(E) = 0, if E is impossible

Example: Suppose a coin is tossed 3 times.

(a) Construct an equiprobable sample space for the experiment.

(b) Find the probability of obtaining 0 heads.

(c) Find the probability of obtaining 2 heads

(a) **{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT},** where HHT indicates that the first two tosses were heads and the third was a tail.

(b) Because there are 8 equally likely possible outcomes, *n*(*S*) = 8. Only one of the eight possible outcomes, *E* = {TTT}, gives 0 heads, so *n*(*E*) = 1. Thus Pr (0 heads) = n(E) / n(S) = 1/8

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(c) The event "two heads" is *F* = {HHT, HTH, THH}, so Pr (2 heads) = n(F) = 3

Odds and Probability

- We sometimes use odds to describe the likelihood that an event will occur. The odds in favor of an event *E* occurring and the odds against *E* occurring are found as follows.

If the probability that event E occurs in Pr(E) =/ 1, then the odds that E will occur are

Odds in favour of E = Pr(E)/1-Pr(E)

The odds that E will not occur, if Pr(E) =/ 0, are Odds against E = 1-Pr(E)/Pr(E)

If the probability of drawing a queen from a deck of playing cards is 1/13, what are the odds
(a) in favor of drawing a queen? (b) against drawing a queen?

(a) 1/13 = 1/13 = 1

1 – 1/13 12/13 = 12 **The odds in favor of drawing a queen are 1 to 12, which we write as 1:12.**

(b) 12/13 = 12

1/13 1 **The odds against drawing a queen are 12 to 1, denoted 12:1.**

Chapter 7.2 (Unions and Intersection of Events: One-Trial Experiments)

If E and F are two events in a sample space S, then the

The **intersection of E and F** is $E \cap F = \{a : a \in E \text{ and } a \in F\}$

The **union of E and F** is $E \cup F = \{a : a \in E \text{ or } a \in F\}$

The **complement of E is** $E' = \{a : a \in S \text{ and } a \notin E\}$

Example

A card is drawn from a box containing 15 cards numbered 1 to 15. What is the probability that the card is (a) even and divisible by 3?

a) If we let *E* represent "even-numbered" and *D* represent "number divisible by 3," we have

E = {2,4,6,8,10,12,14) and D = {3,6,9,12,15}

Mutually Exclusive Events

We say that events *E* and *F* are mutually exclusive if and only **if** $E \cap F = \phi$.

Thus $Pr(E \cup F) = Pr(E) + Pr(F) – 0 = Pr(E) + Pr(F)$

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Example

A firm is considering three possible locations for a new factory. The probability that site A will be selected is 1/3 and the probability that site B will be selected is 1/5.

Only one location will be chosen.

(a) What is the probability that site A or site B will be chosen?

(a) The two events are mutually exclusive, so;

Pr(Site A or Site B) => Pr (Site A $\cup$ Site B)

=> Pr(Site A) + Site(B)

=> 1/3 + 1/5 => **8/15**

Chapter 7.3 (Conditional Probability: The Product Rule)

The **Conditional Probability** that **A** occurs, given that **B** occurs is denoted as **Pr(A|B)**. Is given by: Pr(A|B)=n(A $\cap$ B)/n(B)

The Product Rule

If A and B are probability events, then the probability of event "A and B" is Pr(A and B), and it can be found by one or the other of these two formulas

1) Pr(A and B) = Pr(A $\cap$ B) = Pr(A) x Pr(A|B)

2) Pr(A and B) = Pr(A $\cap$ B) = Pr(B) x Pr(A|B)

Independent Events

The events A and B are **independent** if and only if:

Pr(A|B) = Pr(A) and Pr(B|A) = Pr(B)

Example

A bag contains 4 red marbles, 5 white marbles and 3 black marbles. Find the probability of getting a red marble on the first draw, a black marble on the second draw, and a white marble on the third draw.

a) If the marbles are drawn with replacement

b) If the marbles are drawn without replacement

a) The marbles are replaced after each draw, so the contents are the same on each draw. Thus the events are independent. (Let E_1 be "red on 1st", E_2 be "black on 2nd", E_3 be "white on 3rd"

$Pr(E_1 \cap E_2 \cap E_3) = Pr(E_1) \times Pr(E_2) \times Pr(E_3)$ = > 4/12 x 5/12 x 3/12 = 60/1000 => **15/250**

b) The marbles are not replaced, so the events are independent

$Pr(E_1 \cap E_2 \cap E_3) = Pr(E_1) \times Pr(E_2 | E_1) \times Pr(E_3 | E_1 \text{ and } E_2)$ => 4/12 x 5/11 x 3/9 => **15/297**

Chapter 7.4 (Probability Trees and Bayes Formula)

- Probability trees provide a systematic way to analyze probability experiments that have two or more trials or that use multiple paths within the tree.

Example

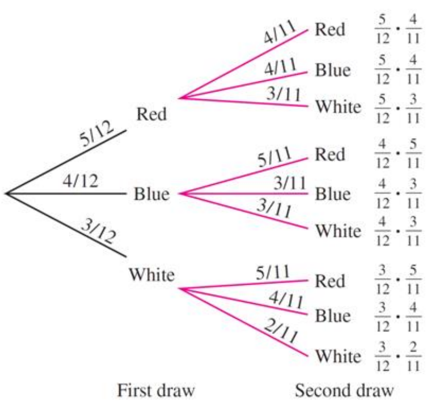
A bag contains 5 red balls, 4 blue balls and 3 white balls. Two balls are drawn one after the other without replacement. Draw a tree representing the experiment and find:

a) Pr(blue on first draw and white on second draw)

b) Pr(white on both draws)

c) Pr(drawing a blue ball and white ball)

d) Pr(second ball is red)



a) Pr(blue on 1st and white on 2nd) = 4/12 x 3/11 => **1/11**

b) Pr(white on 1st and 2nd) = 3/12 x 2/11 = > **1/22**

c) Pr(first Blue, then White) + Pr(first White, then Blue)

=> (4/12 x 3/11) + (2/12 x 4/11) => **2/11**

d) Pr(R, then R) + Pr(B, then R) + Pr(W, then R) => (5/12 x 4/11) + (4/12 x 5/11) + (3/12 x 5/11) => **5/12**

In Bayes problems, we know the result of the second stage of a two-stage experiment and want to find the probability of a specified result in the first stage.

The probability that E_1 occurs in the first stage, given that F_1 has occurred in the second stage,

is $Pr(E_1 | F_1) = Pr(E_1 \cap F_1) / Pr(F_1)$

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Bayes Formula and Trees = $\dfrac{\text{Product of branch probabilities on path leading to } F_1 \text{ through } E_1}{\text{Sum of all branch products on paths leading to } F_1}$

Summary of Probability and Formulas

		One Trial	Two Trials	More Than Two Trials
Pr(*A* and *B*)	Independent	Sample space	Pr(*A*) · Pr(*B*)	Product of probabilities
	Dependent	Sample space	Pr(*A*) · Pr(*B* | *A*)	Product of conditional probabilities
Pr(*A* or *B*)	Mutually exclusive	Pr(*A*) + Pr(*B*)	Probability tree	Probability tree
	Not mutually exclusive	Pr(*A*) + Pr(*B*) − Pr(*A* and *B*)	Probability tree	Probability tree

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